Crashes and Recoveries in Illiquid Markets*

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Abstract

We study the dynamics of liquidity provision by dealers during an asset market crash, described as a temporary, negative shock to outside investors’ aggregate asset demand. We consider a class of dynamic market settings where dealers can trade continuously with each other but trading between dealers and outside investors is subject to delays and involves bargaining. We derive conditions on fundamentals, such as preferences, market structure (dealers’ bargaining strength, the magnitude of trading delays) and the characteristics of the market crash (abruptness, persistence) under which dealers provide liquidity to outside investors following a crash. We also determine the conditions for dealers’ incentives to provide liquidity to be well aligned with society’s interests.

Keywords: liquidity, asset inventories, execution delays, search, bargaining

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1 Introduction

Liquidity in financial markets is warranted by dealers’ willingness to buy or sell assets from their own inventories. While dealers’ liquidity provision might be inconspicuous in normal times, it becomes critical during financial disruptions. In times of crisis it can take a long time for an investor to find a counterpart for trade, either because of technological limitations of order-handling systems and communication networks or, as is the case in over-the-counter markets, because of the decentralized nature of the trading process.\(^1\) These situations appear to be very costly to investors, who concede strikingly large price discounts to unwind their unwanted positions (e.g. the 23% price drop of the Dow Jones Industrial Average on October 19th, 1987). Moreover, the social cost could be even larger than the one directly borne by investors, because of the risk that the financial crisis propagates to the macro-economy (e.g., Borio, 2004). It is commonly believed that, in such circumstances, the liquidity provision of dealers plays a crucial role in mitigating these costs.

This paper studies dealers’ optimal and equilibrium inventory policies during a market crash, described as a temporary, negative shock to outside investors’ aggregate asset demand. We derive conditions such that dealers find it optimal to provide liquidity during the crash. We analyze how dealers’ incentives to provide liquidity change with the structure of the market (e.g., dealers’ bargaining strength or the magnitude of trading frictions) and the characteristics of the crash (e.g., persistence, uncertainty). Lastly, we study conditions for dealers’ incentives to provide liquidity to be well aligned with society’s interests.

We consider a dynamic market setting similar to that of Duffie, Gárleanu, and Pedersen (2005): investors re-balance their holdings in response to changes in their utility for assets, but must engage in a time-consuming process to contact dealers and bargain over the terms of trade.\(^2\) While dealers have no direct utility motive for holding assets, they can trade instantly among each other in a perfectly competitive inter-dealer market. We trigger a crisis with an aggregate negative preference shock to outside investors’ utility for assets, followed by a


\(^2\)There is a recent related literature that focuses on the effects of trading frictions on asset prices. See, for instance, Duffie, Gárleanu, and Pedersen (2006), Gárleanu (2006), Lagos (2006), Longstaff (2005), and Vayanos and Weill (2005). Our analysis is also related to the inventory models of Stoll (1978) and Ho and Stoll (1983). See O’Hara (1997, chapter 2) for a review of this earlier market microstructure literature.
(possibly stochastic) recovery path.\textsuperscript{3} This could represent, for instance, an international shock such as the 1997 Asian crisis and 1998 Russian sovereign default, domestic turbulence such as the September 11 terrorist attack, or even some company specific shocks such as the collapse of Enron.\textsuperscript{4} As shown in Weill (2006), such environments with trading frictions generate a natural role for dealers to provide liquidity in times of crisis. Indeed, dealers find it optimal to accumulate inventories along the recovery path because, although they are not the final asset holders, they can re-allocate assets from early sellers to later buyers more quickly.

Our key departure from previous work is to relax the assumption that investors have linear utility and face an upper bound on their asset holdings. With our general utility and unrestricted holdings, investors’ asset positions become endogenous: this endogeneity turns out to generate new implications for the supply and demand of liquidity. First, as pointed out by Constantinides (1986) in a different context, investors can mitigate the adverse effects of trading frictions by adjusting their asset positions and therefore reducing their demand for liquidity. Second, investors with high utility for the asset can accumulate more asset inventories than their steady-state needs in order to accommodate investors with low utility, just like dealers do. These new effects on the supply and demand of liquidity imply, in contrast to Weill (2006), that whether or not dealers provide liquidity will depend on all the details of market structure, on fundamentals, and on the different dimensions of the crisis.

Our model with endogenous asset holdings has the new implication that the amount of liquidity provided varies non-monotonically with the magnitude of trading frictions. When frictions are small, outside investors choose to take more extreme positions because they know that they can re-balance their asset holdings very quickly. In particular, investors with higher-than-average utility for assets become more willing to hold larger-than-average positions and absorb more of the selling pressure. In some cases, they end up supplying so much liquidity to other investors that dealers don’t find it profitable to step in. If on the contrary trading frictions are large enough, dealers do not accumulate inventories either, but for a different

\textsuperscript{3}Our description of the shock follows the spirit of Grossman and Miller’s (1988) crash dynamics. In Grossman and Miller, however, dealers provide liquidity in order to share risk with outside investors. In our model model, by contrast, dealers have no such utility motive for holding assets: instead, they provide quicker exchange to outside investors. In a related work, Bernardo and Welch (2004) use the feature of non-sequential access of investors to market-makers to describe a market crash as a financial run.

\textsuperscript{4}For a description of how the terrorist attacks of September 2001 disrupted the functioning of financial markets, see the report 03-251 of the General Accounting Office. As an example, failed transactions in the market for government securities rose from around $500 million per day to over $450 billion. The level of fails averaged about $100 billion daily through September 28.
reason: trading frictions reduce investors’ demand for liquidity. Indeed, in order to reduce their exposure to the trading frictions, outside investors choose to take less extreme asset positions and re-balance their holdings by only small amounts. They end up demanding so little liquidity that dealers don’t find it profitable to accumulate inventories. So, if one considers a spectrum of asset markets ranging from very small frictions, such as for large stocks in the New York Stock Exchange (NYSE), to very large trading frictions, such as in the corporate bond market, one would expect to see dealers accumulate more asset inventories during a crash in markets which are in the intermediate range of the spectrum.

From the standpoint of outside investors, an increase in dealers’ bargaining strength is equivalent to an increase in trading frictions. Hence, just as with trading frictions, dealers are less likely to accumulate inventories if their bargaining strength is either small or large. This finding contrasts with the commonly held view that the market power of dealers (e.g., NYSE specialists) is what gives them incentive to provide more liquidity. In our model an increase in bargaining strength may reduce the aggregate amount of inventory accumulated because investors endogenously take less extreme positions and demand less liquidity. Reciprocally, a market reform that reduces dealers’ market power, as observed in equity markets in the 90’s, can raise dealers’ incentives to provide liquidity during a market crash.\footnote{Lagos and Rocheteau (2007) study the effect of market reforms on liquidity in a related search model with endogenous asset positions. However, dealers do not hold inventories and the supply of liquidity is endogenized through a free entry of dealers.}

Our model can also help understand why dealers intervene in some crisis and withdraw in others. In accordance with Hendershott and Seasholes’ (2006) empirical evidence on NYSE specialists inventory strategies, dealers’ incentives to provide liquidity are driven by anticipated capital gains. Hence, dealers are more likely to accumulate inventories when the crisis is severe and expected to be short-lived: a large price drop and the expectation of a quick re-bound make it more profitable for dealers to buy low and sell high.

Finally, we study the social efficiency of dealers’ equilibrium inventory policies. In our setting, every Pareto-optimal allocation maximizes the equally weighted sum of investors’ and dealers’ utility for holding assets, subject to the trading technology. We find that the asset allocation is socially efficient if and only if dealers’ bargaining strength is equal to zero. Given the non-monotonic relationship between aggregate inventory and bargaining strength, this means that dealers may fail to build up inventories in situations where it would be socially efficient to do so, and vice-versa.
From a methodological point of view, this paper builds on Lagos and Rocheteau (2006) and provides a new solution method to analyze search models of financial markets with endogenous asset holdings. In general, models with idiosyncratic (trading) risks are hard to study analytically because the endogenous distribution of asset holdings is an aggregate state that needs to be tracked down along the equilibrium path. The flexibility of our solution method is what allows us to conduct richer comparative dynamics experiments than previous work and to introduce aggregate uncertainty.\footnote{Lagos and Rocheteau (2007) study a similar model which differs in that dealers are not allowed to hold asset inventories.}

The paper is organized as follows. Section 2 lays down the environment. Section 3 characterizes investors’ and dealers’ behavior while Section 4 defines an equilibrium. Section 5 determines the socially optimal allocation. Sections 6 and 7 provide two descriptions of crisis and determine the conditions under which dealers act as providers of liquidity.

2 The environment

Time is continuous and the horizon infinite. There are two types of infinitely-lived agents: a unit measure of investors and a unit measure of dealers. There is one asset and one perishable good which we use as numéraire. The asset is durable, perfectly divisible and in fixed supply $A \in \mathbb{R}_+$. The numéraire good is produced and consumed by all agents. The instantaneous utility function of an investor is $u_i(a) + c$, where $a \in \mathbb{R}_+$ represents the investor’s asset holdings, $c \in \mathbb{R}$ is the net consumption of the numéraire good ($c < 0$ if the investor produces more than he consumes) and $i \in \{1, ..., I\}$ indexes a preference shock. The utility function $u_i(a)$ is, strictly increasing, concave, continuously differentiable and satisfies the Inada condition that $u_i'(0) = 1$. We also assume that it is either bounded below or above. Investors receive idiosyncratic preference shocks that occur with Poisson arrival rate $\delta$. Conditional on the preference shock, the investor draws preference type $i$ with probability $\pi_i$, and $\sum_{i=1}^I \pi_i = 1$. These preference shocks capture the notion that investors value the services provided by the asset differently over time, and will generate a need for investors to periodically change their asset holdings. The instantaneous utility of a dealer is $v(a) + c$, where $v(a)$ is increasing, concave and continuously differentiable.\footnote{One can think of the asset as being a real, durable asset that provides a flow of services to its owner. Also, $a$ could be thought of as shares of a “tree” that yields a real, perishable “fruit” dividend different from the numéraire good. Alternatively, one may adopt the interpretation of Duffie, Gârleanu and Pedersen (2005), and regard $u_i(a)$ as a reduced-form utility function that stands in for the various motives an investor may have for holding the asset, such liquidity or hedging needs. Our qualitative results remain unaffected if we assume that they are smooth and have a common support.}
All agents discount at the same rate $r > 0$.

Figure 1: Trading arrangement

There is a competitive market for the asset. Dealers can continuously trade in this market, while investors can only access the market periodically and indirectly, through a dealer. Specifically, we assume that investors contact a randomly chosen dealer according to a Poisson process with arrival rate $\alpha$. Once the investor and the dealer have made contact, they negotiate the quantity of assets that the dealer will acquire (or sell) in the market on behalf of the investor and the intermediation fee that the investor will pay the dealer for his services. After completing the transaction, the dealer and the investor part ways.\(^8\) The trading arrangement is illustrated in Figure 1.

3 Dealers, investors, and bargaining

In this section we describe the decision problems faced by investors and dealers, and the determination of the terms of trade in bilateral meetings between them. Investors readjust their asset holdings infrequently, at the random times when they meet dealers. In between those times, an investor enjoys the utility flow associated with his current asset position. A dealer’s problem

\(^8\)In actual financial markets, there are position traders who hold asset inventories in the hope of making capital gains. There are also pure spread traders who don’t hold inventories but instead profit exclusively from “buying low and selling high.” Stoll (1978), for example, calls the former dealers and the latter brokers. In our model, the agents that we refer to as dealers engage in both these activities. The analysis would remain unchanged if we were to assume that these activities are carried out by two different types of agents with continuous direct access to the asset market.
consists of continuously managing his own asset position by trading in the asset market. At random times, the dealer contacts an investor who wishes to buy or sell some quantity of assets. At these times, the dealer executes the desired purchase or sale in the asset market on behalf of the investor and receives a fee for his services.\footnote{In principle, the dealer may fill the investor’s order partially or in full by trading out of, or for his own inventory of the asset. For example, if at some time $t$ the dealer contacts an investor who wishes to buy some quantity $a’$ and the dealer’s inventory is $a_d(t) > a’$, then in that instant, the dealer may fill the buy-order by giving the investor $a’$ from his inventory and charging him $p(t) a’$ plus the fee, and instantaneously buying back $a_d(t) - a’$ for his own account in the asset market. Alternatively, the dealer may instead choose not to trade out of his inventory, and simply buy $a’$ in the market on behalf of the investor at cost $p(t) a’$ (and charge him this cost plus the intermediation fee). Clearly, the dealer is indifferent between these modes of execution because he has continuous access to the asset market and all the transactions he is involved in are instantaneous.} We begin with the determination of the terms of trade in bilateral trades between dealers and investors.

Consider a meeting at time $t$ between a dealer who is holding inventory $a_d$ and an investor of type $i$ who is holding inventory $a$. Let $a’$ denote the investor’s post-trade asset holding and $\phi$ be the intermediation fee.\footnote{In our formulation we assume that the investor pays the dealer a fee. However, the bargaining problem can be readily reinterpreted as one in which the dealer pays the investor a bid price which is lower than the market price if the investor wants to sell, and charges an ask price which is higher than the market price if the investor wants to buy. See Lagos and Rocheteau (2006) for details.} The pair $(a’, \phi)$ is taken to be the outcome corresponding to the Nash solution to a bargaining problem where the dealer has bargaining power $\eta \in [0, 1]$. Let $V_i(a,t)$ denote the expected discounted utility of an investor with preference type $i$ who is holding a quantity of asset $a$ at time $t$. Then, the utility of the investor is $V_i(a’,t) - p(t) (a’ - a) - \phi$ if an agreement $(a’, \phi)$ is reached, and $V_i(a,t)$ in case of disagreement. Therefore, the investor’s surplus is $V_i(a’,t) - V_i(a,t) - p(t) (a’ - a) - \phi$. Analogously, let $W(a_d,t)$ denote the maximum attainable expected discounted utility of a dealer who is holding inventory $a_d$ at time $t$. Then, the utility of the dealer is $W(a_d,t) + \phi$ if an agreement $(a’, \phi)$ is reached and $W(a_d,t)$ in case of disagreement, so the dealer’s surplus is equal to the fee, $\phi$.\footnote{The outcome of the bilateral trade does not affect the dealer’s continuation payoff because he has continuous access to the asset market and his trades are executed instantaneously (see footnote 9).} The outcome of the bargaining is given by

$$[a_i(t), \phi_i(a,t)] = \arg \max_{(a’, \phi)} [V_i(a’,t) - V_i(a,t) - p(t) (a’ - a) - \phi]^{1-\eta} \phi^\eta.$$ 

It is clear from this expression that the investor’s new asset holding solves

$$a_i(t) = \arg \max_{a'} [V_i(a’,t) - p(t) a’], \quad (1)$$

and that the intermediation fee is

$$\phi_i(a,t) = \eta \{ V_i[a_i(t), t] - V_i(a,t) - p(t) [a_i(t) - a] \}. \quad (2)$$
Our choice of notation for the bargaining solution in (1) and (2) emphasizes the fact that
the terms of trade depend on the investor’s preference type but are independent of the dealer’s
inventories. In addition, the investor’s post-trade asset holding is independent of his pre-trade
holding, while the intermediation fee is not. From (1) we also see that the investor’s post-trade
asset holding is the one he would have chosen if he were trading in the asset market himself,
rather than through a dealer. According to (2), the intermediation fee is set so as to give the
dealer a share \( \eta \) of the gains associated with readjusting the investor’s asset holdings. These
properties of the Nash bargaining solution can be exploited in writing down the dealer’s and
the investor’s value functions, which we turn to next.

The value function corresponding to a dealer who is holding \( a_d \) shares at time \( t \) satisfies

\[
W(a_d, t) = \sup_{q(s), a_d(s)} \mathbb{E} \left\{ \int_t^T e^{-r(s-t)} \left\{ v[a_d(s)] - p(s)q(s) \right\} ds + e^{-r(T-t)}[\tilde{\phi}(T) + W(a_d(T), T)] \right\},
\]

subject to the law of motion \( a_d'(s) = q(s) \), the short-selling constraint \( a_d(s) \geq 0 \), and the initial
condition \( a_d(t) = a_t \). Here, \( a_d(s) \) represents the stock of assets that the dealer is holding and
\( q(s) \) the quantity that he trades for his own account at time \( s \). The expectations operator, \( \mathbb{E} \),
is taken with respect to \( T \), which denotes the next random time at which the dealer meets an
investor, where \( T - t \) is exponentially distributed with mean \( 1/\alpha \). Since the intermediation fee
determined in a bilateral meeting depends on the investor’s preference type and asset holdings,
and given that the investor is a random draw from the population of investors, at time \( T \) the
dealer expects to extract the average fee

\[
\tilde{\phi}(T) = \int \phi_j(a_i, T) dH_T(j, a_i),
\]

where \( H_T \) denotes the distribution of investors across preference types and asset holdings at time \( T \). The dealer enjoys
flow utility \( v[a_d(s)] \) from carrying inventory \( a_d(s) \), and gets utility \( p(s)q(s) \) from changing this
inventory.

Since intermediation fees are independent of the dealer’s asset holdings, his value function

\[
W(a_t, t) = \max_{q(s)} \left\{ \int_t^\infty e^{-r(s-t)} \left\{ v[a_d(s)] - p(s)q(s) \right\} ds \right\} + \Phi(t), \tag{3}
\]

subject to \( a_d'(s) = q(s) \), \( a_d(s) \geq 0 \) and \( a_d(t) = a_t \). The function \( \Phi(t) \) is the expected
present discounted value of future intermediation fees from time \( t \) onward, and satisfies

\[
\Phi(t) = \mathbb{E}\{e^{-r(T-t)}[\tilde{\phi}(T) + \Phi(T)]\},
\]

where the expectation is with respect to \( T \). This formulation makes it clear that dealers trade assets in two ways: continuously, in the competitive market, or at random times, in bilateral negotiations with investors. Since dealers have quasi-linear preferences
and they can trade instantaneously and continuously in the competitive asset market, their optimal choice of asset holdings is independent from what happens in bilateral negotiations with investors. Next, we analyze the dealer’s inventory accumulation decision.

Consider the optimal control problem in the first term on the right-hand side of (3).

**Lemma 1** Suppose that \( p(t) \) is piecewise continuously differentiable. Then a bounded inventory path, \( a_d(t) \), solves the dealer’s problem if and only if

1. \( p(t) \) does not have any positive jump, \( p(t^+) - p(t^-) > 0 \). When the price has a negative jump,
   \[
   p(t^+) - p(t^-) < 0, \text{ then } a_d(t) = 0;
   \]  
   (4)

2. when \( p(t) \) is differentiable, then
   \[
   v'[a_d(t)] + \dot{p}(t) \leq rp(t) \text{ with an equality if } a_d(t) > 0;
   \]  
   (5)

3. it satisfies the transversality condition
   \[
   \lim_{t \to \infty} e^{-rt} p(t)a_d(t) = 0.
   \]  
   (6)

Note that, because there is a finite measure of assets and agents face a short-selling constraint, dealers’ holdings must be bounded in equilibrium.

The first part of equation (4) means that the price cannot have a positive jump. Otherwise, a dealer could improve his intertemporal utility by buying assets just before the jump, and reselling just after. The opposite trading strategy shows that, when the price jumps down, then the short-selling constraint must be binding. According to (5), whenever a dealer finds it optimal to hold strictly positive inventory, the flow cost of buying the asset, \( rp(t) \), must equal the direct utility flow from holding the asset, \( v'[a_d(t)] \), plus the capital gain \( \dot{p}(t) \). As it is well known from Mangasarian’s results (see Theorem 13, Chapter 3 of Seierstad and Sydsaeter (1987)), together with the other first-order conditions, the transversality condition (6) is sufficient for optimality. Here we show that it is, in fact, necessary.\(^{12}\)

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\(^{12}\)The necessity of such transversality conditions for general formulations of infinite-horizon optimal control problems has been regarded as a delicate issue since Halkin’s (1974) counterexample. See Benveniste and Scheinkman (1982) for fairly general results.
We now proceed with an analysis of an investors’ problem. The value function corresponding to an investor with preference type $i$ who is holding $a$ assets at time $t$, $V_i(a, t)$, satisfies

$$ V_i(a, t) = \mathbb{E}_i \left[ \int_t^T e^{-r(s-t)} u_{k(s)}(a) ds + e^{-r(T-t)} \left\{ V_{k(T)}(a_{k(T)}(T), T) - p(T)(a_{k(T)}(T) - a) - \phi_{k(T)}(a, T) \right\} \right], $$

where $T$ denotes the next time the investor meets a dealer and $k(s) \in \{1, ..., I\}$ denotes the investor’s preference type at time $s$. The expectations operator, $\mathbb{E}_i$, is taken with respect to the random variables $T$ and $k(s)$, and is indexed by $i$ to indicate that the expectation is conditional on $k(t) = i$. Over the interval of time $[t, T]$ the investor holds $a$ assets and enjoys the discounted sum of the utility flows associated with this holding $a$ (the first term on the right-hand side of (7)). The length of this interval of time, $T - t$, is an exponentially distributed random variable with mean $1/\alpha$. The flow utility is indexed by the preference type of the investor, $k(s)$, which follows a compound Poisson process. At time $T$ the investor contacts a random dealer and readjusts his holdings from $a$ to $a_{k(T)}(T)$. In this event the dealer purchases a quantity $a_{k(T)}(T) - a$ of the asset in the market (or sells if this quantity is negative) at price $p(T)$ on behalf of the investor. At this time the investor pays the dealer an intermediation fee $\phi_{k(T)}(a, T)$. Both the fee and the asset price are expressed in terms of the numéraire good.

Substituting the terms of trade (1) and (2) into (7), we get

$$ V_i(a, t) = \mathbb{E}_i \left[ \int_t^T e^{-r(s-t)} u_{k(s)}(a) ds + e^{-r(T-t)} \left\{ (1 - \eta) \max_{a'} \left[ V_{k(T)}(a', T) - p(T)(a' - a) \right] + \eta V_{k(T)}(a, T) \right\} \right]. $$

From the last two terms on the right-hand side of (8), it is apparent that the investor’s payoff is the one he would get in an economy in which he meets dealers according to a Poisson process with arrival rate $\alpha$, and instead of bargaining, he readjusts his asset holdings and extracts the whole surplus with probability $1 - \eta$, whereas with probability $\eta$ he cannot readjust his holdings (and enjoys no gain from trade). Therefore, from the investor’s standpoint, the stochastic trading process and the bargaining solution are payoff-equivalent to an alternative trading mechanism in which the investor has all the bargaining power in bilateral negotiations with dealers, but he only gets to meet dealers according to a Poisson process with arrival rate $\alpha$. 

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\( \kappa = \alpha (1 - \eta) \). Consequently, we can rewrite (8) as

\[
V_i(a, t) = \mathbb{E}_i \left[ \int_t^\tilde{T} u_{k(s)}(a) e^{-r(s-t)} \, ds + e^{-r(\tilde{T}-t)} \{ p(\tilde{T}) a + \max_{a'} [ V_{k(\tilde{T})}(a', \tilde{T}) - p(\tilde{T}) a'] \} \right],
\]  

(9)

where the expectations operator, \( \mathbb{E}_i \), is now taken with respect to the random variables \( \tilde{T} \) and \( k(s) \), where \( \tilde{T} - t \) is exponentially distributed with mean \( 1/\kappa \). From (9), the problem of an investor with preference shock \( i \) who gains access to the market at time \( t \) consists of choosing \( a' \in \mathbb{R}_+ \) in order to maximize

\[
\mathbb{E}_i \left[ \int_t^\tilde{T} e^{-r(s-t)} u_{k(s)}(a') \, ds - \left\{ p(t) - \mathbb{E}_t [ e^{-r(\tilde{T}-t)} p(\tilde{T}) ] \right\} a' \right],
\]  

or equivalently,

\[
\mathbb{E}_i \left[ \int_t^\tilde{T} e^{-r(s-t)} \left\{ u_{k(s)}(a') - [ rp(s) - \dot{p}(s) ] a' \right\} \, ds \right].
\]  

(10)

If the investor had continuous access to the asset market, he would choose his asset holdings so as to continuously maximize \( u_i(a) - [ rp(t) - \dot{p}(t) ] a \), his flow utility net of the flow cost of holding the asset. But since the investor can only trade infrequently, his objective is to maximize (10) instead. Intuitively, the investor chooses his asset holdings in order to maximize the present value of his utility flow net of the present value of the cost of holding the asset from time \( t \) until the next time \( \tilde{T} \) when he can readjust his holdings. The following lemma offers a simpler, equivalent formulation of the investor’s problem.

**Lemma 2** Let

\[
U_i(a) = \frac{(r + \kappa) u_i(a) + \delta \sum_{j=1}^I \pi_j u_j(a)}{r + \delta + \kappa}
\]  

(11)

\[
\xi(t) = (r + \kappa) \left[ p(t) - \kappa \int_0^\infty e^{-(r+\kappa)s} p(t+s) \, ds \right],
\]  

(12)

and assume that \( p(t) e^{-rt} \) is decreasing. Then a bounded process \( a(t) \) solves the investor’s problem if and only if

1. when the investor contacts the market with current type \( i \), his asset holding is \( a(t) = a_i(t) \) with

\[
U_i' \left[ a_i(t) \right] = \xi(t)
\]  

(13)
2. it satisfies the transversality condition

\[ \lim_{t \to \infty} \mathbb{E} \left[ p(\theta_t) a(\theta_t) e^{-r \theta_t} \right] = 0, \]

where \( \theta_t \) denotes the investor’s last contact time with a dealer before \( t \).

Note that assuming decreasing \( p(t) e^{-rt} \) is without loss of generality because it must be true in an equilibrium: it follows from the dealer’s first-order conditions (4) and (5).

Intuitively, \( U_\infty(a) \) is the flow expected utility the investor enjoys from holding \( a \) assets until his next opportunity to re-balance his holdings and \( \xi(t) \) is the cost of buying the asset minus the expected discounted resale value of the asset (expressed in flow terms). Notice that we do not need to know the path for the price of the asset, \( p(t) \), to solve for the investor’s optimal asset holdings. It is sufficient to know \( \xi(t) \). The following lemma establishes the relationship between \( \xi(t) \) and \( p(t) \).

**Lemma 3** Condition (12) implies

\[ rp(t) - \dot{p}(t) = \xi(t) - \frac{\dot{\xi}(t)}{r + \kappa}. \]

To conclude, note that Lemma 3 allows us to rewrite (5) as

\[ v' [a_d(t)] + \frac{\dot{\xi}(t)}{r + \kappa} \leq \xi(t) \text{ with an equality if } a_d(t) > 0 \]

Equations (13) and (16) illustrate the main differences between dealers and investors in our setup. Relative to investors, dealers get an extra return from holding the asset, captured by \( \frac{\dot{\xi}(t)}{r + \kappa} \). This reflects a dealer’s ability to make capital gains by exploiting his continuous access to the asset market. Another difference is the fact that the utility function for investors on the left-hand side of (13) is a weighted-average of the marginal utility flows that the investor enjoys until the next time he is able to readjust his asset holdings.

## 4 Equilibrium

In this section, we study the determination of the asset price, define equilibrium, and show how to characterize it. Since each investor faces the same probability to access the market irrespective of his asset holdings, and since these probabilities are independent across investors, we appeal to the law of large numbers to assert that the flow supply of assets by investors
is $\alpha [A - A_d(t)]$, where $A_d(t)$ is the aggregate stock of assets in dealers’ hands. (Note that $A_d(t) = a_d(t)$ since there is a measure one of identical dealers facing the same strictly concave optimization problem). The measure of investors with preference shock $i$ who are trading in the market at time $t$ is $\alpha n_i(t)$, where $n_i(t)$ is the measure of investors with preference type $i$ at time $t$. Therefore, the investors’ aggregate demand for the asset is $\alpha \sum_i n_i(t)a_i(t)$, and hence the net supply of assets by investors is $\alpha [A - A_d(t) - \sum_i n_i(t)a_i(t)]$. The net demand from dealers is $\dot{A}_d(t)$, the change in their inventories. Therefore, market-clearing requires

$$\dot{A}_d(t) = \alpha \left[ A - A_d(t) - \sum_i n_i(t)a_i(t) \right].$$

The measure $n_i(t)$ satisfies $\dot{n}_i(t) = \delta \pi_i - \delta n_i(t)$ for all $i$, and therefore,

$$n_i(t) = e^{-\delta t}n_i(0) + (1 - e^{-\delta t})\pi_i, \quad \text{for } i = 1, \ldots, I.$$  

If we use (13) to substitute $a_i(t)$ from (17), it becomes apparent that this market-clearing condition determines $\xi(t)$. The intermediation fees along the equilibrium path are given by (2). Using (9), (11) and (12), (2) reduces to

$$\phi_i(a, t) = \eta \left[ U_i(a_i(t)) - U_i(a) - \frac{\xi(t)}{r + \kappa} [a_i(t) - a] \right].$$

**Definition 1** An equilibrium is a collection of bounded asset holdings $[\{a_i(t)\}_{i=1}^I, A_d(t)]$, piecewise continuously differentiable trajectories for prices and intermediation fees, $[\xi(t), p(t), \phi_i(a, t)]$, that satisfies, (4)-(6), (12), (13), (14), (17) and (19).

We do not list the distribution of asset holdings across investors in the preceding definition because it does not affect the dealer’s problem, the investor’s problem, nor any of the variables which are relevant to our analysis. We start by establishing two important properties of any equilibrium price path:

**Lemma 4** In an equilibrium,

$$\lim_{t \to \infty} p(t)e^{-rt} = 0$$

$$p(t) = \int_t^\infty e^{-r(s-t)} \left( \frac{\xi(s) - \frac{\dot{\xi}(s)}{r + \kappa}}{r + \kappa} \right) ds.$$  

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Intuitively, the no-bubble condition (20) follows from adding up the transversality conditions (6) and (14) across all agents of the population. This provides \( \lim_{t \to \infty} \mathbb{E} \left[ p(\theta_t) e^{-r \theta_t} A \right] = 0 \), after observing that agents’ holdings must add up to \( A > 0 \), the aggregate asset supply. This, in turn, can be shown to imply the no-bubble condition (20).

Now, if we combine (13), (16) and (17), and assuming an interior solution for dealers’ inventories, the model can be reduced to a system of two first-order differential equations

\[
\begin{align*}
\hat{A}_d(t) &= \alpha \left\{ A - A_d(t) - \sum_i n_i(t) U_i' \xi(t) \right\}, \quad (22) \\
\dot{\xi}(t) &= (r + \kappa) \left\{ \xi(t) - v' [A_d(t)] \right\}, \quad (23)
\end{align*}
\]

with \( n_i(t) \) given by (18). This system is nonlinear and nonautonomous. The steady-state equilibrium is such that \( U_i'(a_i) = v'(a_d) = \xi = rp \) where \( \xi \) is the unique solution to

\[
\xi' - 1(\xi) + \sum_i \pi_i U_i'^{-1}(\xi) = A. \tag{24}
\]

Consider the limit as the trading frictions vanish, i.e., as \( \alpha \) approaches \( \infty \). From (15), \( \xi(t) = rp(t) - \dot{p}(t) \), so the investor’s cost of investing in the asset is the flow cost \( rp(t) \) minus the capital gain \( \dot{p}(t) \), the same as the dealer’s. From (11), \( U_i(a) \) tends to \( u_i(a) \), so (11) implies that the investor’s optimal choice of assets satisfies \( u_i'(a_i) = rp(t) - \dot{p}(t) \). This is the asset demand of an investor in a frictionless Walrasian market.

A very tractable special case of (22) and (23) obtains when \( n_i = \pi_i \) for all \( i \), i.e., when the distribution of preference types across investors is time-invariant, since the system is then homogenous. (Note that this does not imply that the joint distribution of assets and preference types across investors is constant, so the economy need not be in steady state.) Linearizing (22) and (23) in the neighborhood of the unique steady-state, \( (\tilde{A}_d, \tilde{\xi}) \), it can be verified to be a saddle-point. For some initial condition \( A_d(0) \) in the neighborhood of the steady-state there is a unique trajectory, the saddle-path, that brings the economy to its steady-state. This trajectory also satisfies (6), so it is an equilibrium. Lemma 5 establishes that for a given initial condition, such a path is the unique equilibrium. We represent the dynamics of the system with the phase diagram in Figure 2.

**Lemma 5** Suppose that \( n_i(0) = \pi_i \) for all \( i \), and that the initial condition \( a_d(0) = A_d(0) \) is close to the steady-state value \( \tilde{A}_d \). Then, there is a unique dynamic equilibrium, and it converges to the steady state.
As dealers’ marginal utility for the asset decreases, the $\xi$–isocline shifts downward. As $v'(a_d)$ tends 0, a case we will focus on in the following, the $\xi$–isocline approaches the horizontal axis for all $A_d > 0$ and the vertical axis if $A_d = 0$. The steady state is then at the intersection of the $A_d$–isocline and the vertical axis and there is a saddle-path that brings the economy to the steady state.

5 Efficiency

In this section we derive first-order necessary conditions for an asset allocation to be efficient. We use an elementary variational experiment to identify the social gains associated with dealers holding asset inventories. (We provide a more formal treatment of the social planner’s problem in the appendix). The main result is that an equilibrium allocation is Pareto-optimal if and only if dealers’ bargaining weight, $\eta$, is equal to zero.

In order to study efficient allocations, we define an investor’s marginal value for the asset as

$$M(t, s) \equiv (r + \alpha)E_s \left[ \int_s^T e^{-r(z-s)} m(t, z) \, dz \right].$$

In words, $M(t, s)$ is the (flow) expected present value of an investor’s marginal utility $m(t, \cdot)$ for
the assets he acquired at time \( t \), from time \( s \geq t \) until his next contact time \( T \) with dealers.\(^{13}\) Taking \( \Delta \) to represent the length of a small time interval, the marginal value \( M(t,s) \) solves the recursion

\[
M(t,s) = (r + \alpha)m(t,s)\Delta + (1 - r\Delta - \alpha\Delta)\mathbb{E}_s [M(t,s + \Delta)] . \tag{25}
\]

At each point in time \( t > 0 \), a quantity \( A_d(t) \) of assets is held by dealers, and the remaining quantity, \( A - A_d(t) \), is held by investors. Because our continuum of investors establish contact with dealers at Poisson intensity \( \alpha \), the law of large numbers implies that, during any small time interval \([t, t + \Delta]\), there is a quantity \( A_d(t) + \alpha\Delta [A - A_d(t)] \) of assets that can be reallocated between those investors in contact with dealers, and between investors and dealers.

Holding \( A_d(t) \) fixed, an efficient allocation of the remaining \( \alpha\Delta [A - A_d(t)] \) assets must equate the marginal value \( M(t,t) \) of all investors who are currently contacting dealers, and hold assets. Otherwise one could improve welfare by reallocating assets from some low- to some high-marginal-value investors. This means that,

\[
M(t,t) = \lambda(t), \tag{26}
\]

for some \( \lambda(t) \geq 0 \), the shadow price that the planner assigns to assets in the hands of dealers at time \( t \) (assuming that some investors hold some assets). We now provide a necessary condition for dealers’ inventory holdings, \( A_d(t) \), to be part of an efficient allocation. Precisely, starting from an allocation such that (26) holds at each time, we consider the following perturbed allocation: (i) keep the same allocation during \([0,t]\), (ii) take a marginal asset from some positive measure of “initial” investors at time \( t \) and give them to dealers until time \( t + \Delta \). (iii) If an initial investor recontacts the market at time \( t + \Delta \), give the asset back to him. If he does not recontact the market at time \( t + \Delta \), give the asset to some other “next” investors. (iv) Continue with the initial asset allocation after \( t + \Delta \). Since dealers’ asset holdings at \( t + \Delta \) are the same as in the initial allocation, the quantity of assets available in the market stays the same, and it is feasible to continue with the initial allocation after \( t + \Delta \).

We can break up the net utility of this perturbation as follows. First, during \([t, t + \Delta]\) assets are held by dealers, with a marginal utility \( v'(t) \), instead of the initial investors, with a marginal

\(^{13}\)For instance, in the environment of the previous section, \( m(t,z) = u'_{i(z)} [a_{i(t)}] \), and with \( \eta = 0 \), \( M(t,s) = U'_{i(s)} [a_{i(t)}] \), where \( a_{i(t)} \) is the asset holding chosen by the investor at time \( t \) and \( i(s) \) is the investor’s preference type at time \( s \). We introduce a different notation here so that the present calculations also apply to the environment of Section 7, which features aggregate uncertainty.
utility of \( m(t, t) \).\(^{14}\) This represents a net flow utility of \( (r + \alpha) [v'(t) - m(t, t)] \Delta \). Second, there is a fraction \( \alpha \Delta \) of initial investors who re-establish contact with dealers at \( t + \Delta \) and receive their asset back, with a net utility of zero. For the fraction \( 1 - \alpha \Delta \) of initial investors who do not re-establish contact with dealers, there is an expected discounted cost of

\[
\mathbb{E}_t \left[ e^{-r\Delta} M(t, t + \Delta) \right] \simeq (1 - r\Delta) \mathbb{E}_t [M(t, t + \Delta)].
\] (27)

This represents the discounted marginal value that is lost because, after the transfer and until their respective next contact times with dealers, initial investors hold one unit less of assets. Lastly, because the asset is transferred to some next investors, there is an expected gain of

\[
(1 - r\Delta) \mathbb{E}_t [M(t + \Delta, t + \Delta)].
\] (28)

As before, equation (28) is the discounted marginal value that is gained because, after the transfer and until their respective next contact time with dealers, next investors hold one more unit of assets. This discussion shows that the net utility of the perturbation is

\[
(r + \alpha) [v'(t) - m(t, t)] \Delta + (1 - r\Delta)(1 - \alpha\Delta) \mathbb{E}_t [M(t + \Delta, t + \Delta) - M(t, t + \Delta)].
\] (29)

The second term represents the gain from liquidity provision. The discounting \( (1 - r\Delta) \) means that the gain occurs later in time. The probability \( (1 - \alpha\Delta) \) means that the gain occurs only if the initial investors do not manage to re-establish contact with dealers. The last term \( \mathbb{E}_t [M(t + \Delta, t + \Delta) - M(t, t + \Delta)] \) is positive when the marginal utility \( M(t, t + \Delta) \) of the initial investor is, on average, smaller than the marginal utility \( M(t + \Delta, t + \Delta) \) of the next investor. This means that liquidity provision can raise welfare by improving inter-temporal matching, i.e. creating a mutually beneficial match between two investors who contact dealers at different points in time.

To a first-order approximation, equation (29) can be rearranged as follows\(^{15}\)

\[
(r + \alpha) \Delta \left[ v'(t) - \lambda(t) \right] + (1 - r\Delta - \alpha\Delta) \left\{ \mathbb{E}_t [\lambda(t + \Delta)] - \lambda(t) \right\}.
\] (30)

---

\(^{14}\)Note that, because agents have quasi-linear preference, one should give equal weights to all agents’ marginal utilities for the assets. Indeed, in all Pareto-optimal allocation of assets and numéraire good, the allocation of assets maximizes the equally weighted sum of agents utility for holding assets, subject to the constraints imposed by the trading technology.

\(^{15}\)Use (25) to rewrite (29) as

\[
(r + \alpha)v'(t)\Delta - M(t, t) + (1 - (r + \alpha)\Delta) \mathbb{E}_t [M(t + \Delta, t + \Delta)].
\]

The expression (30) then follows from (26).
Taking $\Delta$ to zero, we obtain that increasing the amount of inventories held by dealers does not improve welfare if

$$v'(t) + \frac{1}{r + \alpha} \lim_{\Delta \to 0} \frac{\mathbb{E}_t [\lambda(t + \Delta) - \lambda(t)]}{\Delta} \leq \lambda(t). \quad (31)$$

Considering the opposite perturbation of decreasing dealers’ inventories, we find that (31) holds with an equality whenever $A_d(t) > 0$.

In the environment of the previous section, with no aggregate uncertainty and where $v'(0) = \infty$, we can derive these first-order conditions formally using the Maximum Principle of optimal control.

**Lemma 6** An efficient allocation $\left\{ a_i(t) \right\}_{i=1}^I, a_d(t)$ satisfies

$$\frac{(r + \alpha) u'_i [a_i(t)] + \delta \sum_j \sigma_j u'_j [a_i(t)]}{r + \alpha + \delta} = \lambda(t), \quad (32)$$

$$v'[a_d(t)] + \frac{\lambda(t)}{r + \alpha} = \lambda(t), \quad (33)$$

the resource constraint (17), and the transversality condition

$$\lim_{t \to \infty} e^{-rt} \lambda(t) = 0, \quad (34)$$

for some $\lambda(t) \geq 0$. In addition, if $a_d(t)$ satisfies

$$\lim_{t \to \infty} e^{-rt} \lambda(t) a_d(t) = 0, \quad (35)$$

then $\left\{ a_i(t) \right\}_{i=1}^I, a_d(t)$ is an optimal path.

If we identify the equilibrium price, $\xi(t)$, with the planner’s shadow price of assets, $\lambda(t)$, and compare (5) and (11) with (32) and (33), it becomes apparent that they would be identical if $\kappa = \alpha$, i.e., if $\eta$ were equal to zero. The following proposition formalizes this observation.

**Proposition 1** Equilibrium is efficient if and only if $\eta = 0$.

Whenever $\eta > 0$, an inefficiency arises from a holdup problem due to ex-post bargaining. When conducting a trade, investors anticipate the fact that they will have to pay fees for rebalancing their asset holdings in the future. These intermediation fees increase with the surplus that the trade generates. As a consequence, investors will tend to avoid positions that could lead in large re-balancing in the future.
6 Crash and deterministic recovery

In this section we describe the dynamic adjustment of the asset price and the allocation of assets between dealers and investors following a market crash. We follow Weill (2006) and think of a “market crash” as a sudden rise in selling pressure and model it as an unexpected shock that modifies the distribution of investors across preference types \( \{n_i(t)\}_{i=1}^I \) in a way that causes the total demand for the asset to fall unexpectedly. We suppose that the economy is in steady state at the time this shock hits. Then, over time, as the distribution of investors across valuations reverts back to the invariant distribution, the selling pressure slowly alleviates and the aggregate demand for the asset eventually returns to normal.

In order to highlight the intermediation role of dealers we assume that dealers start off with no inventory, \( a_d(0) = 0 \), and that they get no utility from holding the asset, i.e., \( v(a) = 0 \). In this formulation, dealers will only buy assets for their own account in an attempt to make capital gains over some holding period. In particular, this implies that \( A_d = 0 \) in the steady state. Indeed, because the price remains constant, a dealer would not make any capital gains when buying and selling, and therefore does not find it optimal to buy assets. For investors we adopt \( u_i(a) = \varepsilon_i a^{1-\sigma}/(1 - \sigma) \), which implies \( U_i(a) = \bar{\varepsilon}_i a^{1-\sigma}/(1 - \sigma) \), with \( \bar{\varepsilon}_i = \frac{(r + \kappa)\varepsilon_i + \delta}{r + \kappa + \delta} \) and \( \bar{\varepsilon} = \sum_k \pi_k \bar{\varepsilon}_k \). The following lemma summarizes the key properties of the investor’s and the dealer’s optimization problems.

**Lemma 7** (a) An investor with preference type \( i \) who gains access to the market at time \( t \) demands

\[
a_i(t) = \left( \frac{\bar{\varepsilon}_i}{\varepsilon_i(t)} \right)^{1/(\sigma)}.
\]

(b) A dealer’s asset holdings satisfy

\[
[r p(t) - \dot{p}(t)] a_d(t) = 0.
\]

The second part of Lemma 7 formalizes the notion that if dealers do not enjoy any direct benefits from holding the asset, then they will only hold it to try to obtain capital gains. Hence, they hold no inventories over periods when the price is growing at a rate lower than the rate of time preference. Conversely, for dealers to be willing to take long positions in the asset, it must be that \( \dot{p}(t)/p(t) = r \). (Naturally, \( \dot{p}(t)/p(t) > r \) would be inconsistent with equilibrium.)
can use (15) to express the dealer’s optimality condition as

\[ \left( \xi(t) - \frac{\dot{\xi}(t)}{r + \kappa} \right) A_d(t) = 0, \tag{38} \]

with $\dot{\xi}(t)/\xi(t) \leq r + \kappa$ and where $A_d(t) \geq 0$ is dealers’ aggregate inventories. (Notice that individual dealers do not need to hold the same inventories.)

With (18) and (36), the market-clearing condition (22) can be written as

\[ \dot{A}_d(t) = \alpha \left\{ A - A_d(t) - \xi(t)^{-1/\sigma} \left[ \bar{E} - e^{-\delta t} \left( \bar{E} - E_0 \right) \right] \right\}, \tag{39} \]

where $\bar{E} = \sum_i \pi_i \bar{\varepsilon}_i^{1/\sigma}$ and $E_0 = \sum_i n_i(0) \bar{\varepsilon}_i^{1/\sigma}$. Intuitively, $\xi(t)^{-1/\sigma} \left[ \bar{E} - e^{-\delta t} \left( \bar{E} - E_0 \right) \right]$ is the total demand of the asset at time $t$ coming from investors. This way of writing the investors’ demand highlights two sources of time variation. First, there is the effective cost of purchasing the asset, $\xi(t)$. The second component, $\left[ \bar{E} - e^{-\delta t} \left( \bar{E} - E_0 \right) \right]$, captures composition effects coming from variations in the distribution of investors over the various preference types. The constant $E$ is a measure of investors’ willingness to hold the asset in the steady state, i.e., when $n_i(t) = \pi_i$, while $E_0$ reflects the investors’ willingness to hold the asset at time 0, when the aggregate shock hits. Thus, $E_0/\bar{E}$ is a measure of the magnitude of the shock to aggregate demand for the asset.

In line with our “market crash” interpretation we maintain $E_0/\bar{E} < 1$ throughout the analysis, i.e., lower preference types receive larger population weights at time 0 relative to the steady state. Thus, aggregate demand for the asset coming from investors is lowest at $t = 0$ when the crisis hits, and then gradually recovers over time as the initial distribution of preference types $\{n_i(0)\}_{i=1}^I$ reverts back to the invariant distribution $\{\pi_i\}_{i=1}^I$.

The dealers’ first-order condition (38) and the market-clearing condition (39) are a pair of differential equations that can be solved for $\xi(t)$ and $A_d(t)$. If $A_d(t) > 0$ for all $t$ in some interval $[t_1, t_2]$, then (38) implies $\xi(t) = e^{(r+\kappa)(t-t_2)} \xi(t_2)$, and given this path for $\xi(t)$, (39) is a first-order differential equation that can be readily solved for the path $A_d(t)$. Similarly, if $A_d(t) = 0$ over some interval, then (39) immediately implies a path for $\xi(t)$. In order to fully characterize the equilibrium path one needs to determine the time intervals over which dealers accumulate inventories as well as the continuity of the trajectory. The following proposition provides the salient features of the equilibrium path following a market crash.

**Proposition 2** The unique equilibrium path $\{\xi(t), A_d(t)\}$ has the following features:
1. It converges to the steady state \( \{ \xi, \tilde{A}_d \} = \{ (\tilde{E}/A)^\sigma, 0 \} \).

2. There exists a time \( T \in [0, \infty) \) such that \( A_d(t) > 0 \) for all \( t \in (0, T) \) and \( A_d(t) = 0 \) for all \( t \geq T \).

3. Let \( p_c(t) \) be the asset price when dealers are constrained to hold no inventory, i.e. \( A_d(t) = 0 \). Then unconstrained dealers will intervene, i.e. \( T > 0 \), if and only if, at the time of the crisis, \( t = 0 \),

\[
\frac{\dot{p}_c(0)}{p_c(0)} > r \Leftrightarrow \frac{E_0}{E} = \frac{\sum_i n_i(0)(r + \kappa)\varepsilon_i + \delta\xi}{\sum_i n_i(0)(r + \kappa)\varepsilon_i + \delta\xi}^{1/\sigma} < \frac{\delta\sigma}{r + \kappa + \delta\sigma}. \tag{40}
\]

According to Proposition 2, the equilibrium path following a market crash is characterized by a switching time \( T \in [0, \infty) \) such that dealers hold the asset for all \( t \in (0, T) \), and do not hold it for \( t \geq T \). It is possible that \( T = 0 \), in which case dealers do not hold inventories at all. If dealers intervene, the time period during which they hold the asset is an interval and starts at the outset of the crisis, i.e., at \( t = 0 \). (See Figure 4 for an illustration.) Thus, dealers never find it beneficial to delay the acquisition of the asset: if they will buy at all, they start buying from the very beginning, when the investors’ selling pressure is strongest. The economic reasoning behind this result is that since dealers get no direct utility from holding the asset, they are only willing to take long positions if the capital gains associated with those positions are large enough, i.e., if the growth rate of the asset price is greater than the discount rate. In the absence of dealers’ intervention, we show that the price \( p_c(t) \) of the asset grows at a decreasing rate, so if dealers don’t have incentives to hold inventories at \( t = 0 \), they never will. This stands in contrast with the result in Weill (2006) according to which dealers do not necessarily start accumulating inventories right after the crash and in fact, for some parameter values, delaying the intervention of dealers is socially optimal.

\[^{16}\text{One key assumption in Weill (2006) is that investor’s utility function is of the Leontief form } u(a) = \min\{a, 1\}. \text{ (Investors there are restricted to hold one or zero units of the asset, while we allow them to hold any nonnegative position.) One can nest Weill’s specification with ours by assuming} \]

\[
u(a) = \left[ \frac{\sigma + (1 - \sigma)a^{\sigma - 1}}{1 - \sigma} \right]^{\theta},
\]

for some \( (\sigma, \theta) \in \mathbb{R}_+ \times (0, 1) \). We obtain our isoelastic utility function in the limit when \( \theta \to 1^- \), and Weill’s Leontief utility function when \( \theta \to 0^+ \). Numerical calculations (not reported here) suggest that, for \( \theta \) close to zero, then one recovers Weill’s result that dealers do not necessarily start accumulating inventories at the time of the crash.

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Lastly, the proposition also formulates condition \( p'(0)/p(0) > r \) in terms of the exogenous severity of the crash, as measured by \( 1 - E_{0}/\bar{E} \), i.e., the magnitude of the initial drop in the investors’ willingness to hold the asset. If this drop is larger than the threshold \( 1 - \frac{\delta r}{r + \kappa + \delta \alpha} \) then dealers will step in to take up the slack resulting from the reduction in the investors’ demand. Conversely, dealers will not intervene if (40) is not satisfied. An equivalent characterization is that dealers will intervene if and only if the rate of growth of the asset price that would result at the time of the crisis if they did not intervene exceeds the rate of time preference. Condition (40) depends on all the fundamentals of the economy, e.g., preferences \((\sigma)\), the extent of trading frictions and the market-power of dealers \((\kappa)\), the change in the distribution of valuations that triggers the crisis \((\{n_i(0)\}_{i=1}^{I})\) and the frequency of the preference shocks \((\delta)\). As shown in the next corollary, there are clearly parametrizations for which this condition does not hold.

**Corollary 1** The set of parameter values under which dealers do not accumulate inventories \((T = 0)\) is not empty.

Corollary 1 is in contrast with the results in Weill (2006) (Theorem 1), where there is always a period of time during which dealers “lean against the wind” before the investors’ selling pressure subsides. A sufficient condition for (40) not to hold is that \((r + \kappa)/\delta\) is sufficiently large. Suppose that preference shocks are very persistent \((\delta\) is very small). In this case the recovery is slow, the growth rate of the asset price is low, and dealers find that the prospective capital gains are smaller than the opportunity cost of holding the asset. It is also instructive to consider the limiting case as \(\alpha\) goes to infinity and the economy approaches the frictionless Walrasian benchmark. In this case dealers no longer have the advantage of trading continuously vis-à-vis investors and their ability to realize capital gains vanishes (recall our discussion of (16)). Put differently, as frictions vanish, the market provides dealers no incentive to buy assets early in the crisis and they do not intervene regardless of the severity of the crisis.

Below we will show that there are also parametrizations for which condition (40) is satisfied and dealers buy assets at the beginning of the crisis, hold them for a while and sell them off as the investors’ selling pressures subside. In these cases dealers choose positive asset positions (foregoing interest on their stock of numéraire goods) even though they get no utility from holding these assets. The reason why dealers may be willing to carry assets is that they have

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\[ ^{17} \text{This difference in the dealers' behavior also arises because investor's asset holdings are unrestricted here but subject to a unit upper bound in Weill (2006).} \]
continuous access to the market while investors do not: this trading advantage allows dealers to "time the market" continuously in order to capture capital gains that investors cannot realize. Without dealers, or if dealers were unable to hold inventories, these capital gains would remain unexploited. In equilibrium, competition among dealers ends up equalizing these capital gains to the opportunity cost of holding assets, i.e., $\frac{\tilde{p}}{p} = r$. This logic is consistent with the frictionless limit we discussed above.

Next, we use numerical examples to illustrate and explain how the key parameters influence the dealers’ incentives to hold inventories in times of crisis. In what we will consider to be the benchmark example, we set $\sigma = 1/2$ and assume that the preference shock can either be $\varepsilon_1 = 0$ or $\varepsilon_2 = 1$, with equal probability. This means that the invariant distribution has an equal measure of investors with low and high valuations. We also set $r = 0.05$ and $\alpha = \delta = 1$, so that on average investors get one preference shock and one chance to trade per “period”. We also set $\eta = 0$ so that the equilibrium of the benchmark parametrization corresponds to the solution to the planner’s problem. We consider an economy which is at its steady state and at time 0 is subject to a shock that causes the fraction of investors with the low preference shock to rise from $n_1(0) = 1/2$ to $n_1(0) = 0.95$.

The shaded (green) regions in Figure 3 illustrate the combinations of parameter values for which condition (40) is satisfied so that dealers hold inventories in times of crisis. In each panel, we let the two parameters in the axes vary and keep the rest fixed at their benchmark values. All panels have $\alpha$—our index of the degree of the trading frictions—on the horizontal axis. Markets with large $\alpha$ are very liquid markets where trades get executed very fast.

Figure 3 allows us to address the following normative question: could it be socially efficient for dealers to accumulate inventories, even though they are “pure speculators" who don’t derive any direct utility from holding assets? The answer is: yes. Recall (Proposition 1) that the equilibrium allocations of an economy with $\eta = 0$ correspond to the Pareto-optimal allocations. Figure 3 (e.g., panel 3) shows that there are parameterizations involving $\eta = 0$ where dealers indeed choose to intervene. As explained in Section 5, the planner allocates assets to dealers in order to exploit an inter-temporal trade-off between the marginal utility of investors in the market at the current date and in the future. The average marginal valuation of the asset across investors is low at the outset of the crisis and higher later on. The planner uses dealers’ inventories to smooth these marginal valuations over time. Specifically, the planner may choose to put assets in the hands of dealers in the earlier stages of the crisis (when the opportunity
cost of not allocating them to investors is relatively low) to be able to transfer these assets without delays to investors in the later stages of the crisis, when the marginal valuation of the average investor is high. Therefore, depending on fundamentals, it can be optimal to have dealers act as a “buffer stock”. The opportunity cost of having dealers carry an asset they don’t value for a while is the price the planner pays to provide immediacy to the future higher-than-average-valuation “cohorts” of investors that will gain access to the asset market at later dates.

Let us now turn to the effects of fundamentals on dealers’ likelihood to intervene during the crisis. We will consider in turn the effects of some characteristics of the crisis, market structure and investors’ preferences.

**Characteristics of the crisis.** The first panel in Figure 3 shows that for any given $\alpha$, dealers intervene if $n_1(0)$ is large enough, i.e., if the crash is sufficiently abrupt. To explain this result we resort to the connection to the planner’s problem (but there is an equivalent explanation

![Figure 3: Parametrizations for which dealers “lean against the wind”](image-url)
in terms of the dealer’s incentives in the equilibrium). In the early stages of the crisis, the “cohorts” of investors that come to contact the marketplace involve a very large fraction of low-valuation investors who have relatively low individual demands for the asset. If the planner chooses not to use the dealers’ inventories, then in these early stages he will be reallocating more assets to the few high-marginal valuation investors. Such an allocation will imply a very low shadow price of assets, denoted $\lambda(t)$ in Section 5, in the early stages of the crisis. Conversely, the shadow price of the asset will be relatively large in later stages as the fraction of high-valuation investors increases toward its steady-state level, since at that point there will be many more high-valuation investors who are willing to hold relatively large quantities of the asset. To larger values of $n_1(0)$ correspond larger discrepancies in the marginal utilities of earlier vis-à-vis later cohorts of traders among which the planner can reallocate assets. This discrepancy is measured by the term $M(t + \Delta, t + \Delta) - M(t, t)$ of (29). Since dealers offer the planner a way to smooth these differences in intertemporal marginal utilities across cohorts, they are used as a buffer stock for large values of $n_1(0)$, i.e., when the crisis is severe.

The second panel in Figure 3 shows that, given $\alpha$, dealers find it optimal to intervene if the recovery is fast enough, i.e. $\delta$ must be large enough. However, the figure also shows that dealers won’t intervene if $\delta$ is too large. This is because $\delta$ not only measures the speed of the recovery, but also the arrival intensity of idiosyncratic preference shocks. With a very large $\delta$, an investor is likely to change type very quickly after trading, and before re-establishing contact with dealers. Because the average type of an investor over his holding period becomes closer to the mean, $\bar{z}$, the economy is then similar to an economy without idiosyncratic preference shocks in which case dealers are not needed to help reallocate assets across time. (See our discussion of Corollary 1).

**Market structure.** We identify the structure of the market with two parameters: $\alpha$, the extent of the trading frictions, and $\eta$, dealers’ bargaining strength.

The first panel in Figure 3 shows that, for a given size of the aggregate shock, dealers provide liquidity if trading frictions are neither too severe nor too small. To understand this effect, consider first the case of a large $\alpha$. In that case, investors anticipate that they can re-balance their asset positions in a fairly short time, $1/\alpha$. This effect increases their willingness to take more extreme position. In particular, investors with higher-than-average utility become more willing to hold larger-than-average positions and absorb more of the selling pressure. In
some cases, when $\alpha$ is large enough, they end up supplying so much liquidity to other investors that dealers don’t find it profitable to step in. If, on the contrary, $\alpha$ is small, then investors behave as if $\varepsilon_i \approx \bar{\varepsilon}$ and they choose asset holdings closer to the mean. The economy is then similar to an economy without idiosyncratic preference shocks, in which case dealers are not needed to help reallocate assets across time.

The third panel in Figure 3 reveals that for any given $\alpha$ dealers are more likely to hold inventories if their bargaining power is neither too large nor too small. Since $\alpha$ and $(1-\eta)$ enter the equilibrium conditions as a product, a large-$\eta$ economy is, from an investor’s standpoint, payoff equivalent to an economy where investors access the market very infrequently, i.e., with a small $\alpha$. Recall that if $\eta = 0$ then the economy is constrained-efficient. Therefore, the third panel also shows that there are parameter values for which dealers intervene in equilibrium although the planner would not have them intervene, and there are also parameter values for which the opposite is true.

**Preferences.** The fourth panel of Figure 3 illustrates the role that $\sigma$, the curvature of the investor’s utility function, plays in the dealer’s decision to hold the asset. First, $\sigma < 1$ is a necessary condition for dealers to intervene. In the case of the most severe crisis possible, i.e., one in which $n_1 (0) = 1$ (no investor values the asset at $t = 0$) one can show that dealers intervene if and only if $\sigma < 1$ regardless of the value of $\alpha$. If $\sigma = 1$ the trajectory of the price is

$$p(t) = \frac{\bar{\varepsilon}}{rA} + \frac{e^{-\delta t}}{(r + \delta)A} \sum_i n_i(0) (\varepsilon_i - \bar{\varepsilon}),$$

which is independent of $\alpha$. In fact, in this case $p(t)$ coincides with the price that would prevail in a frictionless Walrasian market.\textsuperscript{18} But as we argued earlier, in a Walrasian market dealers would hold no assets since arbitrage by investors would prevent the asset price from growing faster than the discount rate. Therefore, dealers never hold inventories for $\sigma = 1$. For lower values of $n_1(0)$, the dealer’s incentives to hold the asset are nonmonotonic in $\sigma$. In particular, for a range of values of $\alpha$ they only hold it if $\sigma$ is in some intermediate range, but not if it is too high or too low. Suppose that $\sigma$ is very large, so that marginal utility is very steep. One could think that there is more room for dealers to smooth differences in intertemporal

\textsuperscript{18}With log preferences an investor’s demand is linear in $\bar{\varepsilon}_i$, so the aggregate demand for the asset only depends on $\bar{\varepsilon}$, i.e., it is independent of $\alpha$. See Proposition 2 in Lagos and Rocheteau (2006). For a related result under a CARA utility function, see Gârleanu (2006). Note also that for $\sigma = 1$, condition (40) reduces to $\sum_i n_i(0) \varepsilon_i < 0$, indicating that dealers never intervene.
marginal utilities across cohorts. However, the gain from liquidity provision in equation (29), 
\[ M(t+\Delta, t+\Delta) - M(t, t+\Delta) \], depends critically on sigma. If \( \sigma \) is very large, then the individual asset demands of high and low valuation investors tend to be closer together, and this reduces the benefit of transferring assets among them. In the extreme case \( \sigma \rightarrow \infty \), \( a_i(t) = A \) for all \( i \) and all \( t \). But of course this means that shocking the invariant distribution from \( \{\pi_i\}_{i=1}^{I} \) to \( \{n_i(0)\}_{i=1}^{I} \) has no effect on asset holdings. So effectively, there is no shock and thus no gain from liquidity provision, even if \( \{\pi_i\}_{i=1}^{I} \) first-order stochastically dominates \( \{n_i(0)\}_{i=1}^{I} \).

Alternatively, one can interpret \( 1/\sigma \) as the elasticity of asset demand, \( a_i \), with respect to the preference shock, \( \varepsilon_i \). As \( \sigma \rightarrow \infty \), asset demand becomes inelastic to the preference shock. In this case, the planner’s shadow price (\( \lambda(t) \) in the notation of Proposition 6) is constant over time, so there is no need nor scope for him to reallocate assets over time.

It is also instructive to look at the opposite extreme of very low \( \sigma \). For example, consider what happens as \( \sigma \rightarrow 0 \) so that investors’ preferences become linear. Suppose that \( \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_I \). From (11) it follows that \( a_i \rightarrow 0 \) for \( i \in \{1,\ldots,I-1\} \), i.e., only investors with the highest marginal utility, \( \varepsilon_I \), hold the asset. Furthermore, \( \xi(t) \rightarrow \bar{\varepsilon}_I \) for all \( t \) and, from (21), \( p(t) \rightarrow \bar{p} = \bar{\varepsilon}_I / r \) for all \( t \). Thus, the price of the asset is constant and equal to its steady-state level. There is clearly no incentive for dealers to buy the asset regardless of the initial shock to the population weights of high and low-type investors. In this extreme case investors’ desired holdings change dramatically in response to preference shocks, but marginal utility is constant at all times among those who demand the asset, therefore a planner would have no need to use dealers to “store” the assets in order to smooth the marginal utilities of cohorts of investors at various points in time.

We can summarize the discussion above as follows. Dealers provide liquidity by accumulating asset inventories if: (i) The market crash is abrupt and the recovery is fast; (ii) Trading frictions are neither too severe nor too small. (iii) Dealers’ market power is not too large; (iv) idiosyncratic preference shocks are not too persistent and investors’ asset demand is not too inelastic with respect to preference shocks.

While Figure 3 illustrates the conditions under which dealers accumulate inventories, it is not informative about the extent of dealers’ intervention, e.g., how much assets do dealers

\footnote{This utility specification is the one used in Duffie, Gârleanu and Pedersen (2005) and Weill (2006). They also assume a unit upper bound on investors’ holdings.}
Figure 4: Dealers’ asset holdings

accumulate, and for how long do they hold them? To answer this question, Figure 4 plots the trajectory for dealers’ inventories for the parameter values of our benchmark example. In both panels one can clearly identify $T$, namely the switching time at which $A_d(t)$ becomes zero after a period over which dealers have held assets.

The first panel illustrates the relationship between market structure, as summarized by $\kappa = \alpha(1 - \eta)$, and dealers’ inventory policy. It reveals that the extent of the trading frictions has a nonmonotonic effect on $T$. The length of the period of time during which dealers hold inventories is first increasing with $\kappa$, because investors take more extreme positions which increases the discrepancy between their marginal utility at different dates. It is decreasing for larger values of $\kappa$ since liquidity is less needed when trading frictions are smaller. The maximum quantity of assets that dealers hold tends to decrease with the extent of the frictions since the measure of investors who contact the market over a small interval of time increases with alpha; so as alpha falls the demand for liquidity is lower.

The second panel of Figure 4 describes dealers’ inventory behavior as a function of the severity of the aggregate shock. First, the holding period shrinks as $n_1(0)$ decreases. Second, the quantity of assets dealers hold at any point in time tends to be larger the more severe the initial reduction in the asset demand. So as the crash is more severe dealers provide more liquidity and for a longer period of time.

The following Proposition compares the trajectory for $\xi(t)$, the effective cost of holding the asset when dealers accumulate inventories, to $\xi_0(t)$, the effective cost of holding the asset when

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28 Together with Proposition 2, Lemma 13, which is stated and proved in the appendix, provides a full characterization of the equilibrium path following a market crash, including closed-form expressions for the paths of $\xi(t)$ and $A_d(t)$. 

28
Proposition 3 If condition (40) holds then there exists $t$ such that $\xi(t) > \xi_0(t)$ for all $t \in [0, t]$ and $\xi(t) < \xi_0(t)$ for all $t \in (t, T)$.

Thus, the presence of dealers mitigates the effect of the market crash on the effective cost of holding the asset. By accumulating inventories right after the shock dealers prevent $\xi$ from falling by too much: $\xi(0)$ is higher than it would have been had dealers not stepped in to buy assets. Figure 5 illustrates Proposition 3.

7 Crash and stochastic recovery

In the previous section our operational definition of a “market crash” was a shock to the distribution of investors across valuations that caused the investors’ total demand for the asset to fall. The recovery path corresponded to the transitional dynamics leading to the steady state, so it was deterministic and it started immediately after the shock. It may be argued that during actual market crashes, the investors’ behavior and the dealers’ decision as of whether to intervene and when to intervene may be affected by uncertainty about the duration of the
crisis. For this reason, in this section we study the dealers’ incentives to provide liquidity in the aftermath of a crisis with an uncertain recovery.

We consider the following scenario. At time zero all investors receive an unanticipated multiplicative shock that temporarily scales down their marginal utility from holding the asset. This constitutes the crash. Subsequently, the economy awaits a “recovery shock” that follows a Poisson process with arrival rate $\lambda$ which causes all investors to simultaneously revert back to their pre-crisis willingness to hold the asset. Formally, we let $T_\lambda$ be an exponentially distributed random variable with mean $1/\lambda$, where $T_\lambda$ denotes the time at which the economy reverts to normal. An investor with preference type $i$ gets utility $u_i(a)$ from holding $a$ for all $t < 0$ and all $t \geq T_\lambda$. For $t \in [0, T_\lambda)$, the investor gets utility $R u_i(a)$, with $R < 1$. Thus, a small $R$ indicates that the crash is severe, and a small $\lambda$ that it is expected to be long-lived.\footnote{One virtue of this formulation is to disentangles the speed $\lambda$ of the recovery and the frequency $\delta$ of the idiosyncratic preference shocks. In the previous section, these were captured by one parameter, $\delta$.} We assume that the stochastic process that describes the recovery is independent of the one that governs an investor’s transitions between preference types. Furthermore, in this section we assume $\{n_i(0)\}_{i=1}^I = \{\pi_i\}_{i=1}^I$, i.e., the initial distribution of preference types is the invariant distribution.

We discuss equilibrium dynamics using Figure 6. (Appendix B provides a detailed solution of the model). We let $A_d^\xi(t)$ be the dealers’ inventories at time $t$ conditional on the $t < T_\lambda$, i.e., the recovery has not occurred before $t$. We denote $\xi^\xi(t)$ the effective cost of holding the asset before the recovery takes place. Similarly, we use the superscript $h$ to indicate variables after the recovery time. The isocline $\dot{A}_d^\xi = 0$ is located to the right of the isocline $\dot{A}_d = 0$ implied by (22). This is because, for any given $\xi$, dealers need to hold more of the asset in order to clear the market. The isocline $\dot{\xi}^\xi = 0$ is downward-sloping and located underneath the saddle-path leading to the long-run steady state, $(\bar{\xi}, 0)$. The equilibrium unfolds as follows. The economy starts at $A_d^\xi(0) = 0$ and at the time of the shock $\xi$ jumps down to the saddle path leading to $(\bar{\xi}^\xi, \bar{A}_d^\xi)$. This saddle path is represented by a dotted line in the figure. The economy then evolves along this saddle-path until the random recovery shock occurs. In the meantime, along this path, dealers’ inventories increase and $\xi^\xi(t)$ decreases. At the random time when the recovery occurs, say $t_\lambda$, the system jumps to the saddle path leading to $(\bar{\xi}, 0)$. This saddle path, denoted $\xi = \psi(A_d)$, is represented by a dashed line in the figure. At the time the recovery shock occurs, the cost of holding the asset jumps from $\xi^\xi$ to $\xi^h$ and dealers start
selling their inventories thereafter until they are completely depleted.

Figure 6: Stochastic recovery: Phase diagram

The following proposition provides a condition under which $A_d'(t) > 0$ for all $t > 0$ before the recovery occurs, i.e., a condition for dealers to “lean against the wind” during a crisis of random duration. It is convenient to define $\tilde{V}_i^\ell(a)$ as the expected sum of discounted utility flows from holding asset $a$ for an investor of preference type $i$ until the next time he contacts a dealer, and $U_i^\ell(a) = (r + \kappa)\tilde{V}_i^\ell(a)$.

**Proposition 4** Let $p_c^h$ be the asset price during the crisis, and $p_c^\ell$ be the price after the stochastic recovery, when dealers are constrained to hold no inventory, i.e. $A_d(t) = 0$. Then, unconstrained dealers will hold inventories during the crisis if and only if:

$$\lambda(p_c^h - p_c^\ell) > r p_c^\ell \iff \sum_i \pi_i U_i^{\ell - 1} \left( \frac{\lambda \xi}{r + \kappa + \lambda} \right) < A. \quad (41)$$

Proposition 4 provides the conditions on fundamentals such that dealers find it beneficial to buy assets during the crisis. As in the case of a deterministic recovery, the condition can be derived by studying prices when dealers are constrained to hold no inventory. If the expected capital gain from holding assets in the event that the economy recovers, $\lambda(p_c^h - p_c^\ell)$, exceeds the flow holding cost, $r p_c^\ell$, then an individual dealers can make profit by purchasing assets during the crisis, and re-sell them when the economy recovers.
In order to establish that (41) does not always hold, we consider two limiting cases. Consider first the frictionless limit \( \alpha \to \infty \). Then, \( U^f_i(a) \to Ru_i(a) \) and the left-hand side of (41) approaches \( \infty \). If investors can access the market as frequently as dealers, there is no role for dealers to provide liquidity by buying assets. Next, consider the case where \( \lambda \to 0 \), i.e., the crisis becomes permanent. Again, it can be checked that the left-hand side of (41) approaches \( \infty \). If the shock is permanent then dealers cannot expect to make capital gains and therefore they do not invest in the asset. We summarize these findings as follows.

**Corollary 2** The set of parameter values under which dealers do not accumulate asset inventories, i.e condition (41) does not hold, is not empty.

Corollary 2 is the analogue of Corollary 1 but for a crisis of random duration. Next, we will show that there are also parameterizations for which condition (41) is satisfied. To this end, let \( u(a) = a^{1-\sigma}/(1 - \sigma) \) and let \( u_i(a) = \varepsilon_i u(a) \). Then, during the crisis, an investor’s flow expected utility \( U^f_i(a) = \hat{\varepsilon}_i u(a) \) where \( \hat{\varepsilon}_i \) is independent of \( a \).

**Corollary 3** Assume \( u_i(a) = \varepsilon_i a^{1-\sigma}/(1 - \sigma) \). Dealers hold inventories during the crisis if and only if

\[
\sum_i \pi_i \hat{\varepsilon}_i^{1/\sigma} < \left( \frac{\lambda}{r + \kappa + \lambda} \right)^{1/2},
\]  

(42)

where

\[
\hat{\varepsilon}_i = \frac{r + \kappa}{r + \kappa + \lambda} \left[ \frac{(r + \kappa + \lambda) \varepsilon_i + \delta \sum_j \varepsilon_j}{r + \kappa + \delta + \lambda} \right] R + \frac{\lambda}{r + \kappa + \lambda} \left( \frac{r + \kappa + \lambda + \varepsilon_i + \delta \sum_j \varepsilon_j}{r + \kappa + \delta + \lambda} \right).
\]

Condition (42) is a condition on fundamentals, including the extent of the trading frictions, preferences and the properties of the aggregate recovery shock. Note that for (42) to hold it is necessary that there are idiosyncratic shocks that create heterogeneity among investors’ positions. To see this, suppose that \( \varepsilon_i = \bar{\varepsilon} \) for all \( i \). Then, (42) reduces to \( 0 > (r + \kappa) R \bar{\varepsilon} \) which is never satisfied. The same would be true if \( \delta \to \infty \). Dealers are useful in this economy if they can reallocate assets among investors with different marginal utilities.

The shaded (green) regions in Figure 7 illustrate the combinations of parameter values for which condition (42) is satisfied so that dealers hold inventories in times of crisis. The benchmark parametrization is: \( \sigma = 0.5, r = 0.05, \pi_1 = \pi_2 = 0.5, \alpha = \delta = 1, \lambda = 1, R = 0.02 \) and \( \eta = 0 \). In each panel, we let the two parameters in the axes vary and keep the rest fixed at their benchmark values. All panels have \( \alpha \)—our index of the degree of the trading frictions—on the horizontal axis.
**Characteristics of the crisis.** The first panel confirms the results in Section 6 that dealers are more likely to accumulate asset inventories when the market crash is severe ($R$ is low). If the shock is severe then dealers can expect larger capital gains when the economy recovers and hence they have higher incentives to accumulate the asset. According to the second panel of Figure 7, for dealers to buy the asset in times of crisis, the crisis must be anticipated to be short-lived ($\lambda$ must be sufficiently high). From the planner’s standpoint if $\lambda$ is low then the opportunity cost of having dealers holding assets (i.e., the foregone utility of current investors) is high. Thus, for $\lambda$ low enough the planner would not use dealers’ inventories to reallocate the asset across investors over time.

**Market structure.** As before, we identify the market structure with the parameters $\alpha$ and $\eta$. The third panel shows that dealers accumulate the asset if their bargaining power is neither too high nor too low. If $\eta$ is close to 1, then investors only enjoy a small gain from re-balancing their asset holdings. As a consequence, when in contact with a dealer they put more weight on their average preferences in order to reduce their need for readjusting their asset holdings in the future. As discussed above, if idiosyncratic preference shocks become less relevant, there is less scope for dealers to reallocate the asset over time. To understand why dealers have lower incentives to provide liquidity when $\eta$ is small, recall that a reduction in $\eta$ is similar from the point of view of agents’ payoffs to an increase in $\alpha$: if trading frictions are reduced then there is less need for the buffer stock of assets provided by dealers.

As illustrated in the first panel of Figure 7, the relationship between dealers’ incentives to accumulate inventories and trading delays is also non-monotonic. Provided that $\alpha$ is not too low, the expected duration of the crisis has to decrease as trading delays become shorter for dealers to provide liquidity. However, if the trading delays are very long ($\alpha$ is close to 0) it is also the case that the crisis has to be short-lived for (42) to be satisfied. When the market is very illiquid, investors hold a quantity of assets which reflects their average preferences for the asset. In this respect, the economy behaves similarly to an economy without idiosyncratic preference shocks ($\varepsilon_i \simeq \bar{\varepsilon}$ for all $i$) in which case dealers are not useful to reallocate the asset over time.

**Preferences.** The fourth panel shows that the curvature of investors’ utility function must be sufficiently small for dealers to accumulate asset inventories. As before, if $\sigma$ is high then
investors’ demand for the asset is relatively inelastic to the idiosyncratic preference shock which reduces the usefulness of dealers.

Lastly, we study how the characteristics of the crash and the structure of the market affect the amount of liquidity provided by dealers. In the Appendix we obtain a closed-form expression for the maximum inventories that dealers are willing to accumulate, \( \bar{A}_d^f = \lim_{t \to \infty} A_d^f(t) \). Assuming (42) holds,

\[
\bar{A}_d^f = A \left( 1 - \Omega \left( \frac{\alpha}{\gamma} + \left( \frac{\gamma - \alpha}{\gamma} \right) \Omega \right)^{\frac{\gamma}{\gamma - \alpha}} \right)
\]  

where \( \gamma \equiv \frac{r + \kappa}{\sigma} \) and \( \Omega \equiv \left( \frac{r + \kappa + \lambda}{\lambda} \right)^{1/\sigma} \sum_i \pi_i (\hat{\varepsilon}_i)^{1/\sigma} / \sum_i \pi_i (\varepsilon_i)^{1/\sigma} \). Notice that the condition (42) is simply \( \Omega < 1 \).

In Figure 8 we plot \( \bar{A}_d^f \) for our baseline parameter values. The first panel confirms the nonmonotonic relationship between dealers’ provision of liquidity and the frictions that prevail.

Figure 7: Parametrizations for which dealers “lean against the wind”

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.01-0.1</td>
<td></td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1-5</td>
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<tr>
<td>( \Omega )</td>
<td>0.1-0.5</td>
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<tr>
<td>( \sigma )</td>
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<tr>
<td>( \kappa )</td>
<td>0.01-0.1</td>
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<tr>
<td>( \lambda )</td>
<td>0.1-1</td>
<td></td>
</tr>
<tr>
<td>( \pi_i )</td>
<td>0.01-0.1</td>
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in the market. The second panel shows that dealers' provision of liquidity increases with the severity of the crisis. However, according to the third panel, the relationship between amount of liquidity provided by dealers and the expected duration of the crisis \((1/\lambda)\) is nonmonotonic. If the shock is very persistent \((\lambda \text{ is low})\) dealers are not willing to accumulate large inventories since the expected capital gain of these inventories is small. If the shock is anticipated to be short-lived \((\lambda \text{ is large})\) dealers accumulate low inventories because the crash reduce investors' asset demand by a small amount. Finally, the fourth panel shows that dealers' inventories decrease as investors' intertemporal elasticity of substitution \((1/\sigma)\) gets smaller. In particular, as investors' utility function gets linear \((\sigma \to 0)\) dealers are willing to accumulate the entire stock of assets in the economy \((\hat{A}_d^\xi \to 1)\).

We can summarize the results in this section as follows. Dealers are more likely to provide liquidity in times of crisis if: (i) The market crash is abrupt and expected to be short-lived; (ii) Dealers' market power is above some minimum value but not too close to one; (iii) Trading delays are neither too long nor too short; (iv) Investors' asset demand is sufficiently elastic to idiosyncratic preference shocks. The provision of liquidity, in terms of the quantity of assets that dealers are willing to accumulate, increases with the severity of the shock but is non-monotonic.
with respect to the duration of the crisis and the extent of the trading frictions.

8 Conclusion

We have studied the equilibrium dynamics of an asset market in the presence of the types of trading delays that are characteristic of many financial markets during times of crisis and of some other markets even in normal times, e.g., over-the-counter markets. In the model, investors are periodically subject to idiosyncratic preference shocks that incite them to reallocate their asset holdings and all trades among investors are intermediated by dealers who can take positions in the asset. We have analyzed the recovery path of the market in the aftermath of an aggregate shock to investors’ preferences which we interpret as a “flight-to-liquidity shock.” Dealers can mitigate the effect of such aggregate shocks on the asset price by stepping in to accumulate asset inventories during the times when investors’ demands collapses. We established conditions on fundamentals, the market structure and the dimensions of the crisis under which dealers find it profitable to provide liquidity during times of market distress. We have also established necessary and sufficient conditions for such an intervention to be efficient.

We showed that dealers are more likely to accumulate asset inventories during a market crash if execution delays are neither too long nor too short. Consequently, a regulation that increases the capacity of dealers to execute a large volume of orders, thereby reducing trading delays, may in fact reduce dealers’ incentives to provide liquidity during a market crash.\(^{22}\) Similarly, dealers are less likely to accumulate inventories in times of crisis if they have a high bargaining power. Hence, a market reform that reduces dealers’ rents can improve liquidity when selling pressures intensify. Finally, since dealers’ incentives to accumulate inventories are based on their expected capital gains, dealers provide more liquidity when the crash is abrupt and short-lived.

From a normative standpoint, we have shown that equilibrium is socially efficient provided that dealers have no market power. Since dealers’ incentives to provide liquidity are non-monotonic with their bargaining strength, there are equilibria where dealers accumulate asset inventories when it is socially inefficient to do so, and there are equilibria for other parameter values where the opposite is true.

\(^{22}\)The regulatory developments in the securities markets since the October 1987 crisis are reviewed in Lindsey and Pecora (1998). According to Lindsey and Pecora (1998, p.290) "most exchanges now have excess capacity of approximately three times that needed for an average trading session."
From a methodological point of view, our paper extends the recent search-theoretic literature on financial markets to allow for unrestricted choices of asset holdings, more general preferences and more general forms of investor heterogeneity, aggregate uncertainty and endogenous provision of liquidity through the dealers’ choice of inventories. These generalizations allowed us to conduct richer comparative statics while maintaining the tractability of the framework.
References


A Proofs

Proof of Lemma 1. Consider any feasible inventory path \( a(t) \). Letting \( t_1 < t_2 \ldots \) be the successive jumps of the price path, and letting \( K \) be the last jump before some time \( t \), we can write, with the convention that \( t_0 = 0 \):

\[
W_0^t(a) = \int_0^t [v(a(z)) - p(z)\dot{a}(z)] e^{-rz} \, dz
\]

\[
= \int_0^t v(a(z)) e^{-rz} \, dz - \sum_{k=1}^K \int_{t_{k-1}}^{t_k} p(z)\dot{a}(z) e^{-rz} \, dz + \int_0^t p(z)\dot{a}(z) e^{-rz} \, dz
\]

\[
= \int_0^t v(a(z)) e^{-rz} \, dz - \sum_{k=1}^K \int_{t_{k-1}}^{t_k} a(z) (rp(z) - \dot{p}(z)) e^{-rz} \, dz - \int_0^t a(z) (rp(z) - \dot{p}(z)) e^{-rz} \, dz
\]

\[
- \sum_{k=1}^K \left[ e^{-rt_k}p(t_k^-)a(t_k) - e^{-rt_k-1}p(t_{k-1}^+)a(t_{k-1}) \right] - \left[ e^{-rt}p(t)a(t) - e^{-rtK}p(t_K^+)a(t_K) \right]
\]

\[
= \int_0^t \left( v(a(z)) - a(z) (rp(z) - \dot{p}(z)) \right) e^{-rz} \, dz + \sum_{k=1}^K a(t_k) e^{-rt_k} \left( p(t_k^+) - p(t_k^-) \right) - e^{-rt}p(t)a(t),
\]

where the second equality follows from integration by part over each interval \([t_{k-1}, t_k]\), and the last equality by collecting time-\( t_k \) terms.

We first establish the “only if” part of the Lemma. Consider any bounded solution \( a(t) \) to the dealer’s problem and suppose that the price has a positive jump up at some \( t_k \). Then, for \( \varepsilon \) small enough, consider the perturbation \( a(t) + \Delta(t) \) where \( \Delta(t) = 0 \) for \( t < t_k - \varepsilon \), \( \Delta(t) = 1 + (t - t_k)/\varepsilon \) for \( t \in [t_k - \varepsilon, t_k] \), \( \Delta(t) = 1 - (t - t_k)/\varepsilon \) for \( t \in [t_k, t_k + \varepsilon] \), and \( \Delta(t) = 0 \) thereafter. Then, using the above, the net utility of this perturbation, \( W_0^\varepsilon(a) - W_0^\infty(a + \Delta) \) is

\[
\int_{t_k-\varepsilon}^{t_k+\varepsilon} \left( v[a(z) + \Delta(z)] - v[a(z)] e^{-rz} - \Delta(z) \left[ rp(z) - \dot{p}(z) \right] \right) \, dz + e^{-rt_k} \left[ p(t_k^+) - p(t_k^-) \right].
\]

Because \( a(z) \) and \( \Delta(z) \) are bounded, the first term goes to zero as \( \varepsilon \) goes to zero, showing that the net utility of the perturbation converges to \( e^{-rt_k} \left[ p(t_k^+) - p(t_k^-) \right] > 0 \), a contradiction, proving that the price can only have a negative jump. If the price has a negative jump \( p(t_k^+) - p(t_k^-) < 0 \) then, as long as \( a(t_k) > 0 \) the reverse perturbation could improve the dealer’s utility. Therefore, if there is a negative jump, then \( a(t_k) = 0 \). Now suppose that, at some differentiability point \( z \), \( v'[a(z)] \geq rp(z) - \dot{p}(z) \). Then, using the expression for \( W_0^t \), one easily shows that a dealer could improve his utility by accumulating more inventories around \( z \). Therefore, \( v'[a(z)] \leq rp(z) - \dot{p}(z) \). If the inequality is strict and \( a(z) > 0 \), then accumulating less inventory around
would improve the dealer’s utility. Therefore, if \( a(z) > 0 \), then \( rp(z) - \hat{p}(z) = 0 \). In order to establish the necessity of the transversality condition (6), we calculate the net-utility of scaling down an optimal path \( a(t) \) by \( 1 - \varepsilon \), for some small \( \varepsilon > 0 \). We find

\[
W_0^t(a) - W_0^t(a(1-\varepsilon)) = \int_0^t \left( v[a(z)] - v[a(z)(1-\varepsilon)] - \varepsilon a(z) [rp(z) - \hat{p}(z)] \right) e^{-r(t-s)} dz - \varepsilon a(t)p(t)e^{-rt}
\]

(44)

Taking the limit \( t \to \infty \) on both sides, we find that

\[
W_0^\infty(a) - W_0^\infty(a(1-\varepsilon)) = \int_0^\infty \left( v[a(z)] - v[a(z)(1-\varepsilon)] - \varepsilon a(z) [rp(z) - \hat{p}(z)] \right) e^{-r(t-s)} dz - \varepsilon \lim_{t \to \infty} a(t)p(t)e^{-rt}.
\]

(45)

Now divide by \( \varepsilon \) and let \( \varepsilon \) go to zero, to find

\[
W_0^\infty(a) - W_0^\infty(a(1-\varepsilon)) = \int_0^\infty \varepsilon a(z) \left( v'[a(z)] - [rp(z) - \hat{p}(z)] \right) e^{-r(t-s)} dz - \lim_{t \to \infty} a(t)p(t)e^{-rt}
\]

\[
= - \lim_{t \to \infty} a(t)p(t)e^{-rt},
\]

(46)

because of the first-order condition (5). (Precise arguments for taking these limits are provided in the last paragraph of the proof.) Because, \( a(t) \) is an optimal path, the net utility calculated above must be positive, meaning that the limit of \( p(t)e^{-rt}a(t) \) must be non-positive. Since \( a(t) \) is positive, \( p(t)e^{-rt}a(t) \) must converge to zero, and we are done. The “if” part of the Lemma follows from Theorem 13, Chapter 3, in in Seierstad and Sydaester (1987).

Lastly, we show that we can take limits in (45) and (46). The left-hand side of (45) converges by definition of the inter-temporal utility. Because of concavity and because of the first-order condition (5), the first term on the right-hand side is positive and increasing, and thus converges to some limit. Now note that \( p(t)e^{-rt} \) is positive and decreasing: indeed it can only jump down and, by the first-order condition (5), its derivative is negative. Hence, \( p(t)e^{-rt} \) is bounded. Because \( a(t) \) is bounded, it follows that \( p(t)e^{-rt}a(t) \) is also bounded. Taken together, this implies that the first-term on the right-hand side of (45) has some finite limit, and so does the second term. For (46), note that

\[
\frac{v[a(z)] - v[a(z)(1-\varepsilon)]}{\varepsilon} - a(z) [rp(z) - \hat{p}(z)] \leq 0
\]

which allows to apply the dominated convergence theorem.

**Proof of Lemma 2.** When evaluating an investor’s inter-temporal utility we can ignore price jumps: this is because the probability that an investor contact the market at a jump time is
equal to zero. We let the random flow utility of an investor at time $t$ be $u(a, t)$, where we use the time argument “$t$” as a short-hand for the investor’s current preference shock.

**Notations.** Considering an individual investor, we let $T_1 < T_2 < \ldots$ be the sequence of his contact times with dealer, with the convention that $T_0 = 0$. Also, we let $N_t$ be the number of contact time during $[0, t]$. Then, for any asset holding plan, $a$, we calculate the inter-temporal utility

$$V_t^0(a) \equiv \int_0^t u(a(z), z) e^{-rz} \, dz - \sum_{n=1}^{N_t} p(T_n) e^{-rT_n} \left[ a(T_n) - a(T_{n-1}) \right],$$

between 0 and $t$, along a realization of the contact time and type processes. On easily shows that this utility can be decomposed as

$$V_t^0 = U_t^0 + B_t^0 + p(T_1)e^{-rT_1}a(0) - p(\theta_t)a(\theta_t)e^{-r\theta_t},$$

where

$$U_t^0(a) = \int_0^t u(a(z), z) e^{-rz} \, dz,$$

$$B_t^0(a) = \sum_{n=1}^{N_t-1} a(T_n) \left[ p(T_n)e^{-rT_n} - p(T_{n+1})e^{-rT_{n+1}} \right].$$

We consider portfolio plans $a$ that are bounded, and such that the intertemporal utility $\mathbb{E}[V_0^\infty(a)]$ is well defined. We first establish:

**Result 1.** $\mathbb{E}[U_0^t(a)]$, $\mathbb{E}[B_0^t(a)]$, and $\mathbb{E}[p(\theta_t)e^{-r\theta_t}a(\theta_t)]$ converge to finite limits.

Let’s start with $\mathbb{E}[U_0^t(a)]$. When the investor’s utility is bounded below, then the result follows from the assumption that the portfolio plan $a$ is bounded. When the investor’s utility is unbounded below and bounded above, we can assume without loss of generality that it is negative. Then $\mathbb{E}[U_0^t]$ is decreasing and thus converge either to some finite or some infinite limit. The limit, in turn, must be finite because

$$\mathbb{E}[U_0^t] = \mathbb{E}[V_0^t] + \mathbb{E}[B_0^t] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)] \geq \mathbb{E}[V_0^t] + \mathbb{E}[B_0^t] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)]$$
$$\geq \mathbb{E}[V_0^t] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)].$$
where the inequality follows because $p(t)e^{-rt}$ is decreasing and $B_0^t$ is therefore positive. Because $\mathbb{E}[V_0^\infty]$ is well defined, the right-hand side of the inequality is bounded below, implying that $\mathbb{E}[U_0^\infty]$ has a finite limit. It then immediately follows that

$$\mathbb{E}\left[ B_0^t + p(\theta_t)e^{-r\theta_t}a(\theta_t) \right] = \mathbb{E}\left[ V_0^t \right] - \mathbb{E}\left[ U_0^t \right] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)],$$

also converges to some finite limit. Note that $B_0^t$ is increasing because $a(t) \geq 0$ and $p(t)e^{-rt}$ is decreasing, implying that $E[B_0^t]$ has a limit. This limit must be finite because the above equality implies that $\mathbb{E}[B_0^t] \leq \mathbb{E}[V_0^t] - \mathbb{E}[U_0^t] - \mathbb{E}[p(T_1)e^{-rT_1}a(0)].$ It then follows that $\mathbb{E}[p(\theta_t)e^{-r\theta_t}a(\theta_t)]$ also has a finite limit, which completes this part of the proof.

**Result 2.** An investor’s inter-temporal utility is

$$\mathbb{E}\left[ V_0^\infty \right] = (r+\kappa)^{-1}\mathbb{E}\left[ \sum_{n=1}^{\infty} e^{-rt}n \left( U(a(T_n), T_n) - \xi(T_n)a(T_n) \right) \right] - \lim_{t \to \infty} \mathbb{E}\left[ p(\theta_t)e^{-r\theta_t}a(\theta_t) \right],$$

where

$$U(a(T_n), T_n) = (r+\kappa)\mathbb{E}\left[ \int_{T_n}^{T_{n+1}} u(a(z), z)e^{-r(z-T_n)} \, dz \mid T_n \right].$$

To show that result, write

$$\mathbb{E}\left[ B_0^\infty \right] = \mathbb{E}\left[ \mathbb{E}\left[ \sum_{n=1}^{\infty} a(T_n) \left[ p(T_n)e^{-rT_n} - p(T_{n+1})e^{-rT_{n+1}} \right] \mid T_n \right] \right] = (r+\kappa)^{-1}\mathbb{E}\left[ \sum_{n=1}^{\infty} a(T_n)\xi(T_n)e^{-rT_n} \right],$$

by definition of $\xi(T_n).$ In addition note that, when $u(\cdot)$ is bounded below, we can without loss of generality assume that it is positive, and we have

$$u[a(z), z]e^{-rt}I_{\{z \leq \theta_t\}} \leq u[a(z), z]e^{-rt}I_{\{z \leq t\}} \leq u[a(z), z],$$

and $u[a(z), z]I_{z \leq \theta_t} \neq u[a(z), z]$ as $t$ goes to infinity. The same reasoning go through with opposite inequalities when $u(\cdot)$ is negative. Therefore, an application of the dominated convergence theorem implies that

$$\mathbb{E}\left[ U_0^\infty \right] = \mathbb{E}\left[ \int_0^{\theta_t} u(a(z), z)e^{-rz} \, dz \right] = \mathbb{E}\left[ \sum_{n=1}^{\infty} \int_{T_n}^{T_{n+1}} u(a(z), z)e^{-rz} \, dz \right]$$

$$= (r+\kappa)\mathbb{E}\left[ \sum_{n=1}^{\infty} e^{-rT_n}U(a(T_n), T_n) \right].$$
where the last equality follows by taking expectations of each term in the sum, with respect to $T_n$.

**Result 3.** The flow inter-contact time utility is $U(a(T_n), T_n) = (r + \kappa)^{-1}U_i(a(T_n))$, where $U_i(a)$ is defined in equation (11) of the Lemma. To see why, denote,

$$\tilde{V}_i(a, t) = \mathbb{E}_i \left[ \int_0^T e^{-rs} u_{k(t+s)}(a') \, ds \Big| k(t) = i \right].$$

Using the Markovian nature of the process $k(t)$, it is easy to see that $\tilde{V}_i(a, t)$ only depends on $t$ through the condition $k(t) = i$ which is already captured by the subscript $i$. Therefore, hereafter we will slightly abuse notation and write $\tilde{V}_i(a)$ for $\tilde{V}_i(a, t)$. Denote $\hat{T}$ the length of the period of time before the investor receives a preference shock. By definition, $\hat{T}$ is exponentially distributed with mean $1/\delta$. The value of an investor can then be written recursively as follows,

$$\tilde{V}_i(a) = \mathbb{E}\left[ \mathbb{I}(\hat{T} < \tilde{T}) \int_0^\hat{T} e^{-rs} u_i(a) \, ds \right] + \mathbb{E}\left[ \mathbb{I}(\hat{T} < \tilde{T}) \int_0^{\hat{T}} e^{-rs} u_i(a) \, ds + \mathbb{I}(\hat{T} < \tilde{T}) e^{-r\hat{T}} \tilde{V}_{k(\hat{T})}(a) \right],$$

(48)

where $k(\hat{T})$ indicates the new realization of the preference shock at time $\hat{T}$. Using the fact that $\tilde{T}$ and $\hat{T}$ are independent random variables, one can rewrite the first term on the right-hand side of (48) as follows:

$$\mathbb{E}\left[ \mathbb{I}(\hat{T} < \tilde{T}) \int_0^\hat{T} e^{-rs} u_i(a) \, ds \right] = \int \int \mathbb{I}(i < \tilde{T}) \kappa e^{-\delta l} e^{-\delta \hat{T}} \int_0^i e^{-rs} u_i(a) \, ds \, d\hat{T} \, dl = \int \kappa e^{-\delta \hat{T}} \int_0^\infty e^{-\delta \hat{T}} \int_0^i e^{-rs} u_i(a) \, ds \, d\hat{T} \, dl = \frac{u_i(a)}{r} \int \kappa e^{-[\kappa + \delta]l} (1 - e^{-rl}) \, d\hat{T} = \frac{\kappa u_i(a)}{(\kappa + \delta)(\kappa + \delta + r)}.$$  

(49)

Similarly, the second term on the right-hand side of (48) can be reexpressed as

$$\mathbb{E}\left[ \mathbb{I}(\hat{T} < \tilde{T}) \int_0^{\hat{T}} e^{-rs} u_i(a) \, ds \right] = \int \int \mathbb{I}(i < \tilde{T}) \kappa e^{-\delta l} e^{-\delta \hat{T}} \int_0^{i} e^{-rs} u_i(a) \, ds \, d\hat{T} \, dl = \frac{\delta u_i(a)}{(\kappa + \delta)(\kappa + \delta + r)}.$$  

(50)
Since the realizations of the preference shocks are independent and identically distributed, the
distribution of \( k(\tilde{T}) \) is given by \( \{\pi_i\}_{i=1}^I \). Therefore,
\[
\mathbb{E} \left[ I_{(T<\tilde{T})} e^{-r(T-\tilde{T})} \tilde{V}_k(T) \right] = \int \int I_{(t<\tilde{T})} \kappa e^{-\kappa t} \delta e^{-\delta t} e^{-rt} dt \int \sum_{k=1}^I \pi_k \tilde{V}_k(a)
\]
\[
= \frac{\delta}{\delta + r + \kappa} \sum_{k=1}^I \pi_k \tilde{V}_k(a). \tag{51}
\]
Adding (49), (50) and (51), one finds
\[
\tilde{V}_i(a) = \frac{u_i(a)}{\kappa + \delta + r} + \frac{\delta}{\delta + r + \kappa} \sum_{k=1}^I \pi_k \tilde{V}_k(a). \tag{52}
\]
After carrying out some calculations (52) yields
\[
\tilde{V}_i(a) = \frac{U_i(a)}{r + \kappa}, \tag{53}
\]
where \( U_i(a) \) is as in (11).

**Result 4.** The expected discounted price at the time the investor regains direct access to the asset market is:
\[
\mathbb{E}[e^{-r(T-\tilde{T})} p(t+\tilde{T})] = \kappa \int_0^\infty e^{-(r+\kappa)s} p(t+s) ds. \tag{54}
\]

**Result 5.** The "only if" part of the Lemma. First, it is clear from (47) that an optimal portfolio strategy should maximize each term \( U(a(T_n), T_n) - \xi(T_n) a(T_n) \), implying the investor's first-order condition. As for the necessity of the transversality condition, consider an optimal asset holding plan and scale it down by \( (1-\varepsilon) \), for some small enough \( \varepsilon \). Using (47), the net change in intertemporal utility can be written
\[
\Delta_\varepsilon = (r + \kappa)^{-1} \left[ \sum_{n=1}^\infty e^{-r T_n} \left[ U(a(T_n), T_n) - U(a(T_n)(1-\varepsilon), T_n) - \varepsilon \xi(T_n) a(T_n) \right] \right] \tag{55}
\]
\[
- \varepsilon \lim_{t \to \infty} \mathbb{E} \left[ a(\theta_t) e^{-r \theta_t} p(\theta_t) \right]. \tag{56}
\]
Now let's divide by \( \varepsilon \). Note that
\[
\frac{1}{\varepsilon} \left[ U(a(T_n), T_n) - U(a(T_n)(1-\varepsilon), T_n) \right] - \xi(T_n) a(T_n) \prec \left( U_a(a(T_n), T_n) - \xi(T_n) \right) a(T_n) = 0,
\]
because of the first-order condition of the Lemma. Convergence is monotonic because of con- 

vexity. This last property allows us to apply the dominated convergence theorem, and we find 

that 

\[ \mathbb{E}\left[ \sum_{n=1}^{\infty} \frac{1}{\varepsilon} \left[ U(a(T_n), T_n) - U(a(T_n)(1 - \varepsilon), T_n) \right] - \xi(T_n)a(T_n) \right] \rightarrow 0, \]

and thus 

\[ \lim_{\varepsilon \to 0} \frac{\Delta \varepsilon}{\varepsilon} = - \lim_{t \to \infty} \mathbb{E}\left[ a(\theta_t)e^{-\theta_t}p(\theta_t) \right] \geq 0. \quad (57) \]

Since \( a(t) \geq 0 \), it follows that \( \lim_{t \to \infty} \mathbb{E}\left[ a(\theta_t)e^{-\theta_t}p(\theta_t) \right] = 0 \), and we are done.

**Result 6.** For the “if” part, we consider a plan \( a \) that satisfies the first-order conditions and 

compare it to some other plan \( a' \). We find

\[
\mathbb{E}[V_0^\infty(a) - V_0^\infty(a')] = \mathbb{E}\left[ \sum_{n=1}^{\infty} e^{-rT_n} \left( U(a(T_n), T_n) - U(a'(T_n), T_n) - \xi(T_n)(a(T_n) - a'(T_n)) \right) \right] + \lim_{t \to \infty} \mathbb{E}\left[ p(\theta_t)e^{-\theta_t}a'(\theta_t) \right] \\
\geq \mathbb{E}\left[ \sum_{n=1}^{\infty} e^{-rT_n} \left( U(a(T_n), T_n) - \xi(T_n) \right) \right] \left( a(T_n) - a'(T_n) \right) \geq 0,
\]

where the first inequality follows because of concavity, and the second inequality follows because 

of the first-order condition of the Lemma and because \( a'(\theta_t) \geq 0 \).

**Proof of Lemma 3.** (a) To obtain (15), rewrite (12) as 

\[ \xi(t) = (r + \kappa)p(t) - \kappa e^{(r+\kappa)t} \int_t^{\infty} (r + \kappa) e^{-(r+\kappa)s} \lambda(s) \lambda(t) ds \quad (58) \]

and differentiate with respect to \( t \).

**Proof of Lemma 4.**

First note that the dealer’s first-order conditions imply that the price can only have negative 

jumps and that \( d/dt(e^{-rt} = \dot{p}(t) = rp(t) \leq 0 \). Hence, \( p(t)e^{-rt} \) is decreasing and positive, and 

thus has a limit. Now we know that 

\[ \mathbb{E}\left[ p(\theta_t)e^{-\theta_t}a(\theta_t) \right] \rightarrow 0, \]
where \( a(t) \) denotes the asset holding of some investor and \( \theta_t \) the last contact time of that investor before \( t \). For the record, note that the cdf of \( \theta_t \) is

\[
\Pr(\theta_t \leq z) = \Pr(N_t - N_z = 0) = e^{-\kappa(t-z)}.
\]

So \( \theta_t \) has an atom at zero, and its cdf is \( \kappa e^{-\kappa(t-z)} \). Another thing we know is that

\[
p(t)e^{-rt}a(t) \to 0,
\]

where \( a(t) \) denote a dealer’s asset holdings. In particular, if one integrates \( p(t)e^{-rt}a(t) \) against the cdf of \( \theta_t \), one finds that

\[
E \left[p(\theta_t)e^{-r\theta_t}a(\theta_t)\right] \to 0,
\]

as \( t \) goes to infinity, because \( \theta_t \) goes to infinity almost surely. Now consider some time \( z \). The sum of asset holdings across investors and dealers must be equal to \( A \), i.e.

\[
\int a^j(z) \, dj = A,
\]

where \( j \) indexes all agents in the economy. Now, we can also write

\[
AE \left[p(\theta_t)e^{-r\theta_t}\right] = E \left[Ap(\theta_t)e^{-r\theta_t}\right] = E \left[\int_j a^j(\theta_t) \, dj \times p(\theta_t)e^{-r\theta_t}\right]
\]

\[
= E \left[\int_j p(\theta_t)e^{-r\theta_t}a^j(\theta_t) \, dj\right] = \int_j E \left[p(\theta_t)e^{-r\theta_t}a^j(\theta_t)\right] \, dj,
\]

As shown above, the last expression goes to zero as \( t \) goes to infinity. Therefore, because \( A > 0 \),

\[
E \left[p(\theta_t)e^{-r\theta_t}\right] \to 0,
\]

as \( t \) goes to infinity. Because we know that \( p(t)e^{-rt} \) converges to some limit, it follows that \( p(t)e^{-rt} \) converges to zero. Indeed, suppose that the limit is strictly positive. Then there is some \( \varepsilon > 0 \) and \( t_\varepsilon \) such that \( p(t)e^{-rt} > \varepsilon \) for all \( t \geq t_\varepsilon \) and

\[
E \left[p(\theta_t)e^{-r\theta_t}\right] \geq E \left[p(\theta_t)e^{-r\theta_t}1_{(t \geq t_\varepsilon)}\right] \geq \varepsilon \Pr(\theta_t \geq t_\varepsilon) = \varepsilon \left(1 - e^{-\kappa(t-t_\varepsilon)}\right) \to \varepsilon
\]

as \( t \) goes to infinity, which is a contradiction.

To arrive at (21), integrate (15) forward using the transversality condition (6).}

**Proof of Lemma 5.** The proof consists of showing that from any initial condition close to the steady state, only the trajectory that follows the saddle path to the steady state is
consistent with individual maximization. Consider Figure 1 and focus on trajectories below the saddle path. These trajectories eventually lead to \( \xi(t) \leq 0 \) or to \( A_d(t) = 0 \). The former are inconsistent with the investor’s optimization (note that (11) would be violated since \( U_i' > 0 \)). The latter are inconsistent with the dealer’s maximization. To see this, integrate (5) forward to obtain

\[
p(t) = e^{rt} \lim_{s \to -\infty} e^{-rs} p(s) + \int_0^\infty e^{-rs} v'[A_d(s + t)] ds.
\]

If we multiply through by \( e^{-rt} \), take limits as \( t \to \infty \), and use the transversality condition (6), this expression implies \( \lim_{t \to \infty} \int_0^\infty e^{-r(s+t)} v'[A_d(s + t)] ds = 0 \), which is violated along trajectories where \( A_d(t) \) equals zero in the limit, or in finite time. Trajectories above the saddle path are also inconsistent with the dealer’s optimization. First, note that \( \xi(t) \) diverges to \( +\infty \) along any such trajectory. From (11), this implies that \( a_i(t) \) converges to zero for each \( i \). In turn, using (17), this implies that \( A_d(t) \) converges to \( A \). Again, (6) and (60) imply \( p(t) = \int_0^\infty e^{-rs} v'[A_d(s + t)] ds \), hence \( \lim_{t \to \infty} p(t) = v'(A)/r \), a constant. But then (12) implies \( \lim_{t \to \infty} \xi(t) = v'(A) < \infty \), i.e., a contradiction that indicates that these paths violate the first-order necessary conditions of the dealer’s problem. Thus, trajectories that lie above the saddle path are not solutions to the dealer’s asset accumulation problem. Conversely, the trajectory that follows the saddle path satisfies the equilibrium conditions (6), (11), (16) and (17), as well as (6).

**Proof of Lemma 6.** We study the problem of a social planner who maximizes the sum of all agents’ utilities, subject to the trading technology. As before, \( H_t(i,a) \) denotes the distribution of investors across preference types and asset holdings at time \( t \). Since at any point in time all investors access the market according to independent stochastic processes with identical distributions, the quantity of assets that the measure \( \alpha \) of randomly-drawn investors make available to the planner is \( \alpha \int adH_t(a,i) = \alpha [A - A_d(t)] \). So the quantity of assets available to be reallocated among agents who are in the market depends on the distribution \( H_t(i,a) \) only through its mean, \( A - A_d(t) \). Consequently, \( H_t(i,a) \) is not a state variable for the planner’s problem. Notwithstanding, in order to allocate assets across investors, the planner needs to know \( n_i(t) = \int 1_{(j=i)} dH_t(j,a) \), i.e., the measure of investors of preference type \( i \) at date \( t \).

Let \( \tilde{V}_i(a) \) denote the expected discounted utility of an investor of type \( i \) who holds a stock of assets \( a \) until the next time his portfolio can be changed, i.e.,

\[
\tilde{V}_i(a) = E_i \left[ \int_t^{t+T} u_k(s)(a)e^{-r(s-t)} ds \right].
\]

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The value function \( \tilde{V}_i(a) \) satisfies

\[
\tilde{V}_i(a) = \frac{(r + \alpha) u_i(a) + \delta \sum_j \pi_j u_j(a)}{(r + \alpha + \delta)(r + \alpha)}.
\] (62)

(The calculations leading to (62) parallel the derivation of \( \tilde{V}_i(a) \) in the proof of Lemma 2.) Since general goods enter linearly in the utility function of all agents, the utilities from production and consumption of those goods net out to 0 and can therefore be ignored by the planner. Thus, the planner only maximizes the direct utilities that dealers and investors enjoy from holding assets. At each date the planner chooses \( q(t) \), the change in the quantity of assets held by dealers and \( a_i(t) \), the quantity of assets allocated to an investor of type \( i \) when he readjusts his portfolio, in order to maximize

\[
\int \tilde{V}_i(a) dH_0(a, i) + \int_0^\infty e^{-rt} \left\{ v[a_d(t)] + \alpha \sum_i n_i(t) \tilde{V}_i[a_i(t)] \right\} dt
\] s.t.

\[
q(t) = \alpha \left[ A - a_d(t) - \sum_i n_i(t) a_i(t) \right],
\] (64)

and subject to the law of motion \( \dot{a}_d(t) = q(t) \), (18), and the initial conditions \( n_i(0) \) and \( a_i(0) \) for \( i = 1, \ldots, I \). The first term in (63) captures the utility of all investors before the first time their portfolios can be reallocated. It is a constant and can therefore be ignored in choosing the optimal allocation. Hence, the planner’s current-value Hamiltonian reduces to

\[
v[a_d(t)] + \alpha \sum_i n_i(t) \tilde{V}_i[a_i(t)] + \mu(t) q(t),
\] (65)

where \( \mu(t) \) is the co-state variable associated with the law of motion for \( a_d(t) \). (The nonnegativity constraints on \( a_i(t) \) and \( a_d(t) \) are slack at all times since \( u_i'(0) = u'(0) = \infty \).) From the Maximum Principle (e.g., Theorem 12 in Seierstad and Sydsæter, 1987), the necessary conditions for an optimum are

\[
\alpha n_i(t) \left\{ \tilde{V}'_i[a_i(t)] - \mu(t) \right\} = 0,
\] (66)

which using (62) can be rewritten as

\[
\frac{(r + \alpha) u_i'[a_i(t)] + \delta \sum_j \pi_j u_j'[a_i(t)]}{r + \alpha + \delta} = (r + \alpha) \mu(t),
\] (67)

and

\[
v'[a_d(t)] + \dot{\mu}(t) = (r + \alpha) \mu(t).
\] (68)
Next, we show that the optimal path must also satisfy the transversality condition
\[
\lim_{t \to \infty} e^{-rt} \mu(t) = 0. \tag{69}
\]

We begin by noticing that for every path of the controls, the functional
\[
U \left[ q(\cdot), \{a_i(\cdot)\}_{i=1}^I \right] = \int_0^\infty e^{-rt} \left\{ v[a_d(t)] + \alpha \sum_i n_i(t) \tilde{V}_i[a_i(t)] \right\} dt + \\
\int_0^\infty e^{-rt} \{ \mu(t) [q(t) - \dot{a}_d(t)] \} dt
\]
with \( a_d(t) = A-q(t)/\alpha - \sum_i n_i(t)a_i(t) \), yields the same value as the planner’s objective function (63) (ignoring the constant term in (63)). Integration by parts implies that
\[
\int_0^\infty e^{-rt} \mu(t) \dot{a}_d(t) dt = e^{-rt} \mu(t) a_d(t) \bigg|_{t=0}^{t=\infty} - \int_0^\infty e^{-rt} [\dot{\mu}(t) - r\mu(t)] a_d(t) dt,
\]
and substituting this expression into (70) yields
\[
U \left[ q(\cdot), \{a_i(\cdot)\}_{i=1}^I \right] = \int_0^\infty e^{-rt} \left\{ v[a_d(t)] + [\dot{\mu}(t) - r\mu(t)] a_d(t) + \alpha \sum_i n_i(t) \tilde{V}_i[a_i(t)] \right\} dt + \\
\int_0^\infty e^{-rt} \mu(t) q(t) dt - e^{-rt} \mu(t) a_d(t) \bigg|_{t=0}^{t=\infty}.
\]

Suppose that \( q(t) \) and \( \{a_i(t)\}_{i=1}^I \) are optimal paths for the controls, then along this optimal trajectory, the implied path for the state variable \( a_d(t) \) is \( A-q(t)/\alpha - \sum_i n_i(t)a_i(t) \). Consider the admissible paths \( \tilde{q}(t,\varepsilon) \) and \( \{\tilde{a}_i(t,\varepsilon)\}_{i=1}^I \), where \( \tilde{q}(t,\varepsilon) = q(t) + \varepsilon \Delta q(t) \) and \( \tilde{a}_i(t,\varepsilon) = a_i(t) + \varepsilon \Delta a_i(t) \), for some arbitrary \( \varepsilon \in \mathbb{R} \). The implied path for the state is \( \tilde{a}_d(t,\varepsilon) = a_d(t) - \varepsilon \Delta a_d(t) \), where \( \Delta a_d(t) = \Delta q(t)/\alpha + \sum_i n_i(t) \Delta a_i(t) \). (An “admissible path” is a path which is piece-wise continuously differentiable and satisfies (64), together with the initial conditions \( \tilde{a}_i(0,\varepsilon) = a_i(0) \) and \( \tilde{a}_d(0,\varepsilon) = a_d(0) \).) Let \( J(\varepsilon) = U \left[ \tilde{q}(\cdot,\varepsilon), \{\tilde{a}_i(\cdot,\varepsilon)\}_{i=1}^I \right] \). Since the paths \( q(t) \) and \( \{a_i(t)\}_{i=1}^I \) are optimal, we must have \( \frac{\partial J(\varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 \), or equivalently,
\[
0 = \int_0^\infty e^{-rt} \left\{ - \left\{ v'[a_d(t)] + \dot{\mu}(t) - r\mu(t) \right\} \Delta a_d(t) + \alpha \sum_i n_i(t) \tilde{V}_i'[a_i(t)] \Delta a_i(t) \right\} dt + \\
\int_0^\infty e^{-rt} \{ \mu(t) \Delta q(t) \} dt + e^{-rt} \mu(t) \Delta a_d(t) \bigg|_{t=0}^{t=\infty}.
\]
If we substitute $\Delta_i(t) = \alpha \Delta_d(t) - \alpha \sum_i n_i(t) \Delta_i(t)$ and notice that $\Delta_d(0) = 0$ (because $\dot{a}_d(t, \varepsilon)$ is admissible), we find that this last expression is equivalent to

$$0 = -\int_0^\infty e^{-rt} \left\{ v'[a_d(t)] + \mu(t) - (r + \alpha) \mu(t) \right\} \Delta_d(t) \, dt + \int_0^\infty e^{-rt} \left[ \sum_i \left( \alpha n_i(t) \left[ V'_i[a_i(t)] - \mu(t) \right] \right) \Delta_i(t) \right] \, dt + \lim_{t \to \infty} e^{-rt} \mu(t) \Delta_d(t).$$

But then (66) and (68) imply that $\lim_{t \to \infty} e^{-rt} \mu(t) \Delta_d(t) = 0$, and since $\Delta_d(t)$ is arbitrary, (69) is a necessary condition for optimality. If we rescale the co-state by defining $\lambda(t) \equiv (r + \alpha) \mu(t)$, it becomes clear that (67), (68) and (69) correspond to (32), (33) and (34), respectively. Finally, the Mangasarian condition (35) is sufficient because the Hamiltonian is jointly concave (see Theorem 13 in Seierstad and Sydsæter, 1987).

**Proof of Proposition 1.** We wish to show that the planner’s optimality conditions and the equilibrium conditions are identical when $\eta = 0$. First, note that the planner’s law of motion (64) is always the same as the market-clearing condition (17). Then let $\lambda(t) = \xi(t)$ and note that the planner’s optimality conditions (32) and (33) are identical to the equilibrium conditions (11) and (5) if and only if $\eta = 0$. To conclude, we must show that (34) is equivalent to (6), but given $\lambda(t) = \xi(t)$, it suffices to show that $\lim_{t \to \infty} e^{-rt} \xi(t) = 0$ if and only if $\lim_{t \to \infty} e^{-rt} \lambda(t) = 0$. From (12),

$$\lim_{t \to \infty} e^{-rt} \xi(t) = \lim_{t \to \infty} e^{-rt} (r + \kappa) \int_0^\infty e^{-(r+s)t} \left\{ rp(t) - \kappa [p(t+s) - p(t)] \right\} \, ds = (r + \kappa) \int_0^\infty e^{-(r+s)t} \left\{ r \lim_{t \to \infty} e^{-rt} p(t) - \kappa \lim_{t \to \infty} e^{-rt} [p(t+s) - p(t)] \right\} \, ds = r \lim_{t \to \infty} e^{-rt} p(t).$$

**Proof of Lemma 7.** For part (a), note that the investor’s asset demand (36) is immediate from (11) given the functional form assumptions. The Hamiltonian corresponding to the dealer’s problem is $-p(t) q(t) + \rho(t) q(t) + \zeta(t) a_d(t)$, where $\rho(t) \geq 0$ is the costate variable and $\zeta(t) \geq 0$ is the multiplier on the constraint $a_d(t) \geq 0$. The Maximum Principle (e.g., Theorem 12 in Seierstad and Sydsæter, 1987) delivers $\rho(t) = p(t)$ and $\zeta(t) = r p(t) - \dot{p}(t)$, together with the complementary slackness condition $\zeta(t) a_d(t) = 0$. This implies $[r p(t) - \dot{p}(t)] a_d(t) = 0$, which together with the fact that $r p(t) - \dot{p}(t) \geq 0$ implies part (b).
Before proving Proposition 2, we establish several intermediate results (Lemmas 8–13) which will aid us in the proofs.

We begin with a characterization of the equilibrium trajectories of \( A_d(t) \) and \( \xi(t) \) over arbitrary time-intervals:

**Lemma 8** (i) Consider a time-interval \([t_1, t_2]\) such that \( A_d(t) > 0 \) for all \( t \in (t_1, t_2) \). Then,

\[
A_d(t) = \alpha \left\{ \frac{A^{1-e^{-\alpha t}}}{\alpha} - \frac{E}{e^{-(r+\kappa)t_2} \xi(t_2)} \right\}^{1/\sigma} e^{-\frac{r+\kappa}{\sigma} t} + \frac{E-E_0}{e^{-(r+\kappa)t_2} \xi(t_2)} \left\{ \frac{1-e^{-\alpha \frac{r+\kappa}{\sigma} t}}{\alpha - \frac{r+\kappa}{\sigma}} \right\} e^{-\frac{r+\kappa}{\sigma} t} \tag{71}
\]

and \( \xi(t) = \xi^+(t) \), where

\[
\xi^+(t) = e^{(r+\kappa)(t-t_2)} \xi^+(t_2) \tag{72}
\]

for all \( t \in (t_1, t_2) \).

(ii) Consider a time-interval during which \( A_d(t) = 0 \). Then, \( \xi(t) = \xi_0(t) \) for all \( t \) in such interval, where

\[
\xi_0(t) = \left\{ 1 - \frac{r + \kappa}{r + \kappa + \delta \sigma} e^{-\delta (t-t)} \right\}^\sigma \xi, \tag{73}
\]

with \( \delta = (1/\sigma) \ln \left[ \frac{r+\kappa+\delta \sigma}{r+\kappa} \left( 1 - \frac{E_0}{E} \right) \right] \).

**Proof.** (i) Consider an interval \((t_1, t_2)\) such that \( A_d(t) > 0 \) for all \( t \) in that interval. From (38), \( \dot{\xi}(t) / \xi(t) = r + \kappa \) which gives (72). Substituting this expression into (39), implies that \( A_d(t) \) satisfies

\[
\dot{A}_d(t) + \alpha A_d(t) = \alpha A - \frac{E e^{-\frac{r+\kappa}{\sigma} t}}{e^{-(r+\kappa)t_2} \xi(t_2)} \left( \frac{1-e^{-\alpha \frac{r+\kappa}{\sigma} t}}{\alpha - \frac{r+\kappa}{\sigma}} \right) e^{-\frac{r+\kappa}{\sigma} t},
\]

and (71) is the solution to this first-order differential equation. In the case of resonance where \( \frac{r+\kappa}{\sigma} = \alpha \), the solution becomes

\[
A_d(t) = \alpha \left\{ \frac{A^{1-e^{-\alpha t}}}{\alpha} - \frac{E}{e^{-(r+\kappa)t_1} \xi(t_1)} e^{-\alpha t} - \frac{E-E_0}{e^{-(r+\kappa)t_1} \xi(t_1)} \frac{1-e^{-\alpha \delta t}}{\delta} e^{-\delta \xi(t_1)} \right\}.
\]

There is a second nongeneric case of resonance where \( \frac{r+\kappa+\delta \sigma}{\sigma} = \alpha \). In this case, the solution becomes

\[
A_d(t) = \alpha \left\{ \frac{A^{1-e^{-\alpha t}}}{\alpha} - \frac{E}{e^{-(r+\kappa)t_1} \xi(t_1)} e^{-\alpha t} + \frac{E-E_0}{e^{-(r+\kappa)t_1} \xi(t_1)} e^{-\alpha t} \right\}.
\]
To avoid repetitive derivations, we restrict our analysis to the generic case, where \( \alpha - \frac{r + \kappa}{\sigma} \neq 0 \) and \( \alpha - \frac{r + \kappa + \delta}{\sigma} \neq 0 \).

(ii) Consider a time interval \((t_1, t_2)\) such that \(A_d(t) = 0\). From (39), \(A_d(t) = \dot{A_d}(t) = 0\) implies \(\xi(t) = \xi_0(t)\) with \(\xi_0(t)\) given by (73). ■

The following lemma establishes a key continuity property of equilibrium prices and allocations.

**Lemma 9** In any equilibrium, \(A_d(t)\) and \(\xi(t)\) are continuous for all \(t\).

**Proof.** To establish the continuity of \(A_d(t)\) we proceed in three steps. (i) From Lemma 8 it is immediate that \(A_d(t)\) and \(\xi(t)\) are both continuous on every open interval \((t_1, t_2)\) over which \(A_d(t) > 0\) for all \(t \in (t_1, t_2)\) or \(A_d(t) = 0\) for all \(t \in (t_1, t_2)\). (ii) We establish that if \(A_d(t) > 0\) for all \(t \in (t_1, t_2)\), and \(A_d(t) = 0\) for all \(t \in (t_2, t_3)\), then \(A_d(t)\) must be continuous at \(t_2\). Assume this is not the case, i.e., suppose that \(\lim_{t \uparrow t_2} A_d(t) > 0\), but \(A_d(t_2) = 0\). If dealers are reducing their asset holdings discretely at \(t_2\), by market clearing, it must be that the investors who are in the market at \(t_2\) are increasing their holdings discretely. But since their demands are continuous decreasing functions of \(\xi(t)\), this can only happen if \(\xi(t)\) has a downward jump at \(t_2\). (Since there is only a measure 0 of investors in the market at any point in time, investors’ demand would have to be infinite at \(t_2\) and \(\xi(t_2) = 0\).) Rearranging (58) from the proof of Lemma 3, we get

\[
p(t) - \frac{\xi(t)}{r + \kappa} = \kappa e^{(r + \kappa)t} \int_t^\infty e^{-(r + \kappa)s} p(s) ds.
\]

Thus, since the right-hand side is continuous in \(t\), any pointwise downward jump in \(\xi(t)\) corresponds a pointwise downward jump in \(p(t)\). Since \(\lim_{t \uparrow t_2} A_d(t) > 0\), we have \(\lim_{t \uparrow t_2} a_d(t) > 0\) for at least some dealer(s). Focus on any such dealer’s problem as \(t_2\) approaches. In the proposed equilibrium, \(p(t_2^-) - p(t_2) > 0\), and \(a_d(t_2^-) - a_d(t_2) = a_d(t_2^-) > 0\), so in the interval \((t_2^-, t_2)\), the dealer’s utility from trading inventories is \(p(t_2) a_d(t_2^-)\), the proceeds of his asset sale at \(t_2\) (recall that, \(\dot{p}(t)/p(t) = r\) while \(a_d(t) > 0\), so he is getting zero utility from trading inventories on \((t_1, t_2)\)). But this dealer could have attained a payoff \(p(t_2^+) a_d(t_2^+) > p(t_2) a_d(t_2^-)\) by selling off his inventory an instant before the price jumped downward. Thus, we conclude that the equilibrium path \(A_d(t)\) cannot exhibit this type of discontinuity. In this part we have considered the case where the discontinuity is from the left, i.e., \(\lim_{t \uparrow t_2} A_d(t) > A_d(t_2) = 0\). The case where \(\lim_{t \uparrow t_2} A_d(t) = A_d(t_2) > \lim_{t \downarrow t_2} A_d(t) = 0\) is handled similarly. (iii) By an
Lemma 4. and A continuous for all held by investors in the market is show that the continuity of jumping upward.) Together, steps (i)–(iii) imply that any equilibrium path \( A_d(t) \) must be continuous for all \( t \). To conclude, we establish that \( \xi(t) \) must be continuous for all \( t \). First, we show that the continuity of \( A_d(t) \) implies that \( \xi(t) \) cannot have a downward jump at \( t_2 \). The continuity of \( A_d(t) \) means that \( A_d(t_2) = 0 \), which together with the nonnegativity constraint \( A_d(t) \geq 0 \) implies \( \dot{A}_d(t_2^-) \leq 0 \leq \dot{A}_d(t_2^+) \). Since \( A_d(t_2^+) = A_d(t_2^-) = 0 \), (39) yields

\[
\left[ \frac{1}{\xi(t_2^+)} - \frac{1}{\xi(t_2^-)} \right] = 0.
\]

and therefore, \( \xi(t_2^-) \leq \xi(t_2^+) \). The fact that \( \xi(t) \) cannot have an upward jump, i.e., that \( \xi(t_2^-) < \xi(t_2^+) \) cannot be part of an equilibrium, follows from a no-arbitrage argument like the one in step (iii). Hence, \( \xi(t_2^-) = \xi(t_2^+) \). ■

The following lemma shows that there is no equilibrium in which dealers hold positive inventories at all dates.

**Lemma 10** There is no equilibrium with \( A_d(t) > 0 \) for all \( t < \infty \).

**Proof.** Otherwise, it follows from the dealer’s first-order condition that \( rp(t) = \dot{p}(t) \) and therefore that \( p(t) e^{-rt} = p(0) \). Since \( p(0) > 0 \), this violates the no-bubble condition (20) of Lemma 4. ■

Lemma 11 shows that the \( \tilde{t} \) defined in part (c) of the statement of Proposition 13 has the property that dealers will hold inventories for all \( t < \tilde{t} \).

**Lemma 11** In any equilibrium, \( \{ t : t \leq \tilde{t} \} \subseteq \{ t : A_d(t) > 0 \} \), where \( \tilde{t} = \ln \left[ \frac{r + \kappa + \delta \sigma}{r + \kappa} (1 - \frac{E_0}{E}) \right]^{1/\delta} \).

**Proof.** Suppose the contrary, i.e., that \( A_d(t) = 0 \) for all \( t \in (t', t'') \), with \( t'' < \tilde{t} \). Then \( \xi(t) = \left[ 1 - \frac{r + \kappa + \delta \sigma}{r + \kappa} e^{-\delta(t-t')} \right]^\sigma \) for all \( t \in (t', t'') \) (by part (ii) of Lemma 8). Thus,

\[
\frac{\dot{\xi}(t)}{\xi(t)} = \frac{\delta \sigma (r + \kappa)}{(r + \kappa + \delta \sigma) e^{-\delta(t-t') - (r + \kappa)} - (r + \kappa)}
\]

for all \( t \in (t', t'') \). But note that \( \dot{\xi}(t) / \xi(t) > r + \kappa \) for all \( t < \tilde{t} \), so the proposed path for \( A_d(t) \) violates the dealer’s first-order condition (38) on \( (t', t'') \). ■
Lemma 12 establishes that the equilibrium asset holdings of dealers after a crash follow a very precise pattern: if dealers hold positive inventories, they will do so from the outset of the crash, over a connected interval of time of finite length \( \tilde{t} \), and will hold no inventories thereafter.

**Lemma 12** In any equilibrium, \( \{ t : A_d(t) > 0 \} = [0, T) \) where \( 0 \leq T < \infty \).

**Proof.** We first show that if \( A_d(t') = 0 \), then \( A_d(t) = 0 \) for all \( t \geq t' \). (Note that this immediately implies that \( \{ t : A_d(t) > 0 \} = [0, \tilde{t}) \), with \( \tilde{t} \geq 0 \) but possibly infinite.) We proceed by contradiction. Suppose that \( A_d(t) \) is part of an equilibrium, with \( A_d(t) = 0 \) for all \( t \in (t' - \Delta^-, t'] \) and \( A_d(t) > 0 \) for all \( t \in (t', t' + \Delta^+) \), for some \( \Delta^- \), \( \Delta^+ > 0 \). Then, from (73) (part (ii) of Lemma 8), \( \xi(t) = \xi_0(t) \) for all \( t \in (t' - \Delta^-, t'] \), where \( \xi_0(t) = \left[ 1 - \frac{r + \kappa}{r + \kappa + \delta \sigma} e^{-\delta(t - \tilde{t})} \right]^\sigma \xi \), and from (72) (part (i) of Lemma 8), \( \xi(t) = \xi^+(t) \) for all \( t \in (t', t' + \Delta^+) \), where \( \xi^+(t) = e^{(\kappa + \delta)(t - \tilde{t})} \xi^+(t') \). From Lemma 9 we know that \( \xi(t) \) must be continuous, so \( \xi^+(t) = e^{(\kappa + \delta)(t - \tilde{t})} \xi_0(t') \) on \( (t', t' + \Delta^+) \). From Lemma 11 we know that for \( A_d(t) = 0 \) on \( t \in (t' - \Delta^-, t'] \) to be part of an equilibrium, it must be that \( t' > t' - \Delta^- \geq \tilde{t} \), so \( \hat{\xi}_0(t)/\xi_0(t) = \frac{\delta \sigma (r + \kappa)}{(r + \kappa + \delta \sigma) e^{-\delta(t - \tilde{t})} - (r + \kappa)} \leq r + \kappa \) for all \( t \geq t' - \Delta^- \) (with strict inequality for \( t > t' - \Delta^- \)). But then the fact that \( \xi_0(t') = \xi^+(t') \) and \( \hat{\xi}_0(t)/\xi_0(t) < r + \kappa = \xi^+(t) / \xi^+(t) \) for all \( t > t' \) implies that \( \xi^+(t) > \xi_0(t) \) for all \( t > t' \). Since \( \xi(t) \) must be continuous, this would imply an equilibrium with \( A_d(t) > 0 \) for all \( t > t' \). But this is a contradiction, since we know by Lemma 10 that such a path for \( A_d(t) \) is inconsistent with the dealer’s transversality condition. Thus, if dealers hold inventories at all in equilibrium, they must do so from \( t = 0 \) and for an uninterrupted period of time, up to some time \( T \geq 0 \). Finally, the fact that \( T < \infty \) follows by appealing to Lemma 10 once again. Figure 5 illustrates the main idea of this proof. \( \blacksquare \)

**Lemma 13** Following a market crash:

(a). If dealers do not intervene, the equilibrium is \( A_d(t) = 0 \) and \( \xi(t) = \xi_0(t) \) for all \( t \), with

\[
\xi_0(t) = \left[ 1 - \frac{r + \kappa}{r + \kappa + \delta \sigma} e^{-\delta(t - \tilde{t})} \right]^\sigma \xi
\]

and \( \tilde{t} = (1/\delta) \ln \left[ \frac{r + \kappa + \delta \sigma}{r + \kappa} \left( 1 - \frac{E_0}{E} \right) \right] \).

(b). If dealers intervene, the equilibrium is

\[
\xi(t) = \begin{cases} 
\xi^+(t) & \text{for } t < T \\
\xi_0(t) & \text{for } t \geq T
\end{cases} \quad \text{ and } \quad A_d(t) = \begin{cases} 
A_d^+(t) & \text{for } t < T \\
0 & \text{for } t \geq T
\end{cases}
\]

55
where \( \xi_+(t) = e^{(r+\kappa)(t-T)}\xi_0(T) \),

\[
A_d^+(t) = \alpha \left\{ \frac{1-e^{-\alpha t}}{\alpha} + \frac{e^{-r\kappa t} e^{-\delta(T-t)}}{r + \kappa + \delta \sigma} \left( \frac{1-e^{-\alpha(T-t)}}{\alpha} \right) e^{-\delta(t-i)} \right\} A,
\]

and \( T \geq \hat{t} \) is the unique positive root of

\[
\int_0^T e^{\alpha s} \left[ 1 - e^{-\frac{r+\kappa}{r+\kappa+\delta \sigma}(T-s)} \right] ds = 0.
\]

**Proof of Lemma 13 and Proposition 2.** From Lemma 12, we know that an equilibrium must have \( A_d(t) > 0 \) for all \( t \in [0, T) \) and \( A_d(t) = 0 \) for \( t \geq T \), with \( 0 \leq T < \infty \), so we construct such an equilibrium to establish parts (a) and (b) of Lemma 13. For part (a), note that \( A_d(t) = 0 \) and \( \xi(t) = \xi_0(t) \) for \( t \geq T \) by part (ii) of Lemma 8. Thus in particular, this is true if \( T = 0 \) (i.e., if dealers do not intervene). For part (b), note that again, \( A_d(t) = 0 \) and \( \xi(t) = \xi_0(t) \) for \( t \geq T \). For \( t < T \), we have \( A_d(t) \) and \( \xi(t) = \xi^+(t) \), given by (71) and (72), respectively, in the proof of Lemma 8. Since \( \xi(t) \) must be continuous (Lemma 9), \( \xi^+(T) = \xi_0(T) \), so \( \xi^+(t) = e^{(r+\kappa)(t-T)}\xi_0(T) \), which in the statement of Lemma 13 is denoted \( \xi_+(t) \). The expression for \( A_d(t) \) for \( t < T \), i.e., (71), reduces to \( A_d^+(t) \) in the statement of Lemma 13 after setting \( t_2 = T \) and \( \xi(T) = \xi_0(T) = \left[ 1 - \frac{r+\kappa}{r+\kappa+\delta \sigma} e^{-\delta(T-t)} \right]^\sigma \xi \), using \( E_0/E = 1 - \frac{r+\kappa}{r+\kappa+\delta \sigma} e^{\delta t} \), and rearranging terms. So far we have described the full equilibrium for a given switching date \( T \).

To determine \( T \), we use the fact that \( A_d(t) \) must be continuous, which implies \( A_d(T) = 0 \), a condition to be solved for \( T \). To derive this condition, we start with (39), which leads to

\[
\int_0^t e^{\alpha s} \left[ \hat{A}_d(s) + \alpha A_d(s) \right] ds = \alpha \int_0^t e^{\alpha s} \left\{ A - \xi(s)^{-1/\sigma} \left[ \hat{E} - e^{-\delta s} (\hat{E} - E_0) \right] \right\} ds
\]

and in turn to

\[
A_d(t) = \alpha \int_0^t e^{-\alpha(t-s)} \left\{ A - \frac{\hat{E}}{\xi(s)^{1/\sigma}} \left[ 1 - \frac{r+\kappa}{r+\kappa+\delta \sigma} e^{-\delta(s-i)} \right] \right\} ds.
\] (74)

For \( t \leq T \), \( \xi(s) = \xi_+(s) = e^{(r+\kappa)(s-T)}\xi_0(T) = e^{(r+\kappa)(s-T)} \left[ 1 - \frac{r+\kappa}{r+\kappa+\delta \sigma} e^{-\delta(T-t)} \right]^{\sigma} \xi \), and substituting this into (74) yields

\[
A_d(t) = \alpha A \int_0^t e^{-\alpha(t-s)} \left[ 1 - e^{-\frac{r+\kappa}{r+\kappa+\delta \sigma}(T-s)} \frac{1-e^{-\delta(s-i)} e^{-\delta(T-t)}}{1-e^{-\delta(T-t)}} \right] ds.
\] (75)
Thus, \( A_d(T) = 0 \) if and only if \( \Gamma(T) = 0 \), where
\[
\Gamma(T) = \int_0^T e^{as} \left[ 1 - e^{-\frac{r + \kappa}{\sigma}(T-s)} \frac{1 - \frac{r + \kappa}{r + \kappa + \delta} e^{-\delta(s-t)}}{1 - \frac{r + \kappa}{r + \kappa + \delta} e^{-\delta(T-t)}} \right] ds.
\]
This is the same map we used to define \( T \) in the statement of Lemma 13, so the proof of part \( (b) \) of the lemma is complete. Finding \( T \) reduces to finding the zeroes of the map \( \Gamma \). Note that \( \Gamma(0) = 0 \) and \( \Gamma(T) \to -\infty \) as \( T \to \infty \), so \( \lim_{T \to 0} \Gamma'(T) > 0 \) is sufficient to guarantee the existence of some \( T \in (0, \infty) \) such that \( \Gamma(T) = 0 \) (i.e., dealers intervene). If in addition, we can show that \( \Gamma'(T) < 0 \) for \( T > 0 \), then this will guarantee that the root is unique. Conversely, if \( \lim_{T \to 0} \Gamma'(T) \leq 0 \), then \( \Gamma'(T) < 0 \) for \( T > 0 \) implies there exists no \( T > 0 \) such that \( \Gamma(T) = 0 \) (i.e., dealers do not intervene). Differentiating, we find
\[
\Gamma'(T) = \frac{r + \kappa}{\alpha \sigma} \left[ \frac{t + \kappa}{r + \kappa + \delta} \left( e^{aT} - 1 \right) - \frac{t + \kappa}{r + \kappa + \delta} \left( e^{aT} - 1 \right) \right].
\]
From Lemma 11 we know that \( T \geq \hat{t} \), so for \( T > 0 \), \( \Gamma'(T) \) has the same sign as
\[
-\left[ 1 - e^{-\delta(T-t)} \right],
\]
which is negative. As \( T \to 0 \), this expression is positive if and only if \( \hat{t} > 0 \), which amounts to condition (40) in the statement of Proposition 2. Hence, a \( T > 0 \) such that \( A_d(T) = 0 \) exists (i.e., dealers intervene) if and only if (40) holds, and when such a \( T \) exists, it is unique. To link this condition to \( \hat{p}_c(t)/p_c(t) \), recall that \( \hat{p}(t)/p(t) > r \) if and only if \( \check{\xi}(t)/\xi(t) > r + \kappa \) (e.g., from (15)). Then, from (39), if dealers do not intervene, \( \check{\xi}(t)/\xi(t) = \frac{\delta \sigma(r + \kappa)}{(r + \kappa + \delta \sigma) e^{-\delta(t-\hat{t})} - (r + \kappa)} \), which is decreasing in \( t \) and equal to \( r + \kappa \) at \( \hat{t} = (1/\delta) \ln \left[ \frac{r + \kappa + \delta \sigma}{r + \kappa} \left( 1 - E_0/\bar{E} \right) \right] \). Thus, \( \lim_{t \to 0} \check{\xi}(t)/\xi(t) = \frac{\delta \sigma(r + \kappa)}{(r + \kappa + \delta \sigma) e^{-\delta(t-\hat{t})} - (r + \kappa)} > r + \kappa \iff \hat{t} > 0 \), and this last condition is equivalent to (40). Finally, notice that the uniqueness of the equilibrium follows from Lemma 12 and the uniqueness of the switching time \( T \) such that \( \Gamma(T) = 0 \). The convergence to the steady state is immediate from the equilibrium prices and allocations described in parts \( (a) \) and \( (b) \) of Lemma 13.

**Proof of Proposition 3.** First, note that from parts \( (c) \) and \( (d) \) of Proposition 13, \( \xi(s) - \xi_0(s) \geq 0 \) if and only if
\[
1 - e^{-\frac{r + \kappa}{\sigma}(T-s)} \frac{1 - \frac{r + \kappa}{r + \kappa + \delta} e^{-\delta(s-t)}}{1 - \frac{r + \kappa}{r + \kappa + \delta} e^{-\delta(T-t)}} \geq 0.
\]
We first establish that \( \xi_0(0^+) < \xi(0^+) \). We proceed by contradiction. Suppose \( \xi_0(0^+) \geq \xi(0^+) \). From the proof of Lemma 12 we know that \( \check{\xi}_0(t)/\xi_0(t) \) is decreasing, with \( \check{\xi}_0(t)/\xi_0(t) \to 0 \) as
\( t \to \infty \). In addition, under condition (40), \( \dot{\xi}_0(t)/\xi_0(t) > r + \kappa \) at \( t = 0^+ \). Therefore, there is a unique \( T > 0 \) such that \( \xi_0(T) = e^{(r+\kappa)T} \xi(0^+) \). For all \( s \in (0, T) \), \( \xi_0(s) > \xi(s) = \xi_+(s) \) and therefore

\[
1 - e^{\frac{r+\kappa}{\sigma}(T-s)} \cdot \frac{1 - \frac{r+\kappa}{r+\kappa+\sigma} e^{-\delta(s-t)}}{1 - \frac{r+\kappa}{r+\kappa+\sigma} e^{-\delta(T-t)}} < 0,
\]

which together with (75) implies \( A_d(t) < 0 \) for all \( t \in (0, T) \), a contradiction. Thus, \( \xi_0(0^+) < \xi(0^+) \). Finally, the fact that \( \ln \xi_0(t) < \ln \xi(0^+) \), \( \ln \xi_0(T) = \ln \xi(T) \) and that there is a \( \hat{t} \in (0, T] \) defined as in Proposition 13 such that \( \frac{d}{dt} \ln \xi_0(t) \geq r + \kappa = \frac{d}{dt} \ln \xi(t) \) if and only if \( t \in [0, \hat{t}] \), implies there is a unique \( \xi(t) > \xi_0(t) \) for all \( t \in (0, \hat{t}) \) and \( \xi(t) < \xi_0(t) \) for all \( t \in (\hat{t}, T) \). See Figure 5 for an illustration.

\section*{B Stochastic recovery}

Suppose the recovery occurs at some time \( t_\lambda \), i.e., \( t_\lambda \) is the realization of the random variable \( T_\lambda \). We begin by describing the equilibrium of the economy after the recovery, taking as given dealers’ inventories at the time the recovery occurs, \( A_d(t_\lambda) \). Once we have solved for the equilibrium from the time of the recovery onward, we solve for the equilibrium price and allocations before the recovery and piece both sets of paths together to characterize the full equilibrium from the outset of the crash at \( t = 0 \).

Consider first the economy after the recovery. Let \( V^h_i(a, t, t_\lambda) \) denote the value function corresponding to an investor who has preference type \( i \) and is holding portfolio \( a \) at time \( t \), conditional on the recovery having occurred at time \( t_\lambda \leq t \). The investor’s value function is

\[
V^h_i(a, t, t_\lambda) = \mathbb{E}_t \left\{ \int_t^{\hat{t}} u_k(s)(a) e^{-r(s-t)} ds + e^{-r(\hat{t}-t)} \{ p^h(\hat{t}, t_\lambda)a + \max_{a'} [V^h_k(a', \hat{t}, t_\lambda) - p^h(\hat{t}, t_\lambda)a'] \} \right\},
\]

(76)

where \( p^h(t, t_\lambda) \) denotes the asset price. Notice that (76) is identical to (9) except for the fact that \( p^h(\hat{t}, t_\lambda) \) replaces \( p(\hat{t}) \). Therefore, the investor’s problem is the same as in Lemma 2 where \( p^h(t, t_\lambda) \) replaces \( p(t) \). The dealer solves

\[
W^h(a_d, t, t_\lambda) = \max_{q(s)} \int_t^\infty -e^{-r(s-t)} p^h(s, t_\lambda) q(s) ds
\]

subject to \( \dot{a}_d(s) = q(s) \), \( a_d(s) \geq 0 \) for all \( s \geq t \), and the initial condition \( a_d(t_\lambda) = a_d \). The problem (77) is analogous to (3).
For all $t \geq t_\lambda$, the equilibrium is characterized by the pair of differential equations (22) and (38), together with the initial condition $A_d(t_\lambda)$. The following lemma characterizes the equilibrium path that the economy follows after the recovery has taken place.

**Lemma 14** Suppose the economy recovers at some time $t_\lambda \geq 0$. Then, there exists a unique equilibrium path $\{\xi(t), A_d^b(t)\}$ for $t \geq t_\lambda$ such that:

(a) For all $t \in (t_\lambda, T)$,

$$
\xi(t) = \bar{\xi} e^{-(r+\kappa)(T-t)}
$$

(78)

$$
A_d^b(t) = e^{-\alpha(t-t_\lambda)} A_d(t_\lambda) + \alpha \int_{t_\lambda}^{t} e^{-\alpha(t-s)} \left[ A - \sum_i \pi_i U_i^{-1} [\xi(s)] \right] ds,
$$

(79)

where $T < \infty$ is the unique solution to $A_d^b(T) = 0$.

(b) For all $t \geq T$, \( \{\xi(t), A_d^b(t)\} = (\bar{\xi}, 0) \), where $\bar{\xi}$ solves $\sum_i \pi_i U_i^{-1}(\bar{\xi}) = A$.

**Proof.** Note that if $A_d(T) = 0$ for some $T \geq t_\lambda$, then (22) and (38) imply $\{\xi(t), A_d(t)\} = (\bar{\xi}, 0)$ for all $t \geq T$. Thus, let $T = \inf \{t \geq t_\lambda : A_d(t) = 0\}$. Next we show that $T < \infty$ by establishing that $A_d(t) > 0$ for all $t \geq t_\lambda$ is inconsistent with equilibrium. Note that if $A_d(t) > 0$ for all $t \geq t_\lambda$, (22) and (38) imply, after a change of variable,

$$
A_d(t) = e^{-\alpha(t-t_\lambda)} A_d(t_\lambda) + \alpha \int_{0}^{t-t_\lambda} e^{-\alpha u} \left[ A - \sum_i \pi_i U_i^{-1} [\xi(t-u)] \right] du
$$

with $\xi(s) = e^{(r+\kappa)(s-t_\lambda)} \xi(t_\lambda)$. Thus, $\lim_{t \to \infty} A_d(t) = A > 0$. From (37), $p(t) = e^{r(t-t_\lambda)} p(t_\lambda)$ which implies $\lim_{t \to \infty} e^{-rt} p(t) = e^{-rt_\lambda} p(t_\lambda) > 0$. The dealer’s transversality condition is violated, so $A_d(t) > 0$ for all $t \geq t_\lambda$ cannot be part of an equilibrium. We conclude that $T < \infty$ and this establishes part (b) of the lemma. For part (a), first note that the same arguments we used in Lemma 9 can be applied here to establish that $\xi(t)$ and $A_d(t)$ are continuous for all $t \in (t_\lambda, \infty)$. (The only difference is that Lemma 9 is proven with $U_i(a) = \bar{\xi}^a \frac{1}{1-a}$, but this is immaterial for the results.) In particular, this means that $\xi(t)$ and $A_d(t)$ are continuous at $t = T > t_\lambda$. For any $t \in (t', t'') \subset (t_\lambda, T]$, (22) and (38) imply that $\{\xi(t), A_d(t)\}$ are given by (79) and (78), where (78) uses $\xi(T) = \bar{\xi}$, which follows from the continuity of $\xi(\cdot)$. We use the continuity of $A_d(\cdot)$, to determine $T$: using (79) and (78), $A_d(T) = 0$ can be written as

$$
A_d(t_\lambda) + \alpha \int_{t_\lambda}^{T} e^{\alpha(s-t_\lambda)} \left[ A - \sum_i \pi_i U_i^{-1} \left[ e^{(r+\kappa)(s-T)} \bar{\xi} \right] \right] ds = 0.
$$

(80)
The left-hand side of (80) is equal to \( A_d(t_\lambda) \geq 0 \) at \( T = t_\lambda \) and goes to \(-\infty\) as \( T \to \infty \). Differentiate the left-hand side of (80) with respect to \( T \) to get:

\[
\alpha(r + \kappa) \int_{t_\lambda}^{T} e^{\alpha(s-t_\lambda)} \sum_i \pi_i \frac{e^{(r+\kappa)(s-T)} \xi}{U_i^r [\tilde{a}_i(s)]} ds < 0,
\]

where \( \tilde{a}_i(s) = U_i^{r-1} [e^{-(r+\kappa)(T-s)}] \). So there is a unique \( T \) that satisfies \( A_d(T) = 0 \). To conclude, the uniqueness of the equilibrium follows from the fact that the saddle path leading to the steady state depicted in Figure 9 is the only path that satisfies all the equilibrium conditions. Any other path is inconsistent with the dealer’s optimization: paths above the saddle path violate the transversality condition while those below would imply an upward jump in \( \xi(t) \) at \( t = T \) (see Figure 9).

According to Lemma 14 the equilibrium path of the economy starting from \( t_\lambda \) is such that \( A_d(t) > 0 \) for all \( t \) in the interval \( (t_\lambda, T) \) and \( A_d(t) = 0 \) for all \( t \geq T \). Furthermore, \( T > t_\lambda \) unless \( A_d(t_\lambda) = 0 \). According to (78), the investor’s effective cost of holding the asset, \( \xi(t) \), increases at rate \( r + \kappa \) while dealers hold inventories, meanwhile according to (79), the stock of assets held by dealers decreases monotonically until it is fully depleted at time \( T \). (To see this, notice from (78) that \( \xi(t) < \tilde{\xi} \) for all \( t < T \). As a consequence, \( A - \sum_i \pi_i U_i^{r-1} [\xi(t)] < 0 \) for all \( t < T \) and from (22) \( \dot{A}_d(t) < 0 \).) The condition \( A_d(T) = 0 \) can be rewritten as

\[
A_d(t_\lambda) + \alpha \int_0^{T-t_\lambda} e^{\alpha s} \left\{ A - \sum_i \pi_i U_i^{r-1} \left[ e^{-(r+\kappa)(T-t_\lambda)-s} \right] \right\} ds = 0. \tag{81}
\]

From (81) the time that it takes for dealers’ inventories to be depleted, \( T - t_\lambda \), is an implicit function of the stock of inventories in dealers’ hands at the recovery time, \( A_d(t_\lambda) \). Equivalently, (24) provides a relationship between the effective cost of holding the asset at the recovery time, \( \xi(t_\lambda) = \tilde{\xi} e^{-(r+\kappa)(T-t_\lambda)} \), and dealers’ initial inventories, \( A_d(t_\lambda) \). We represent this relationship by the function \( \psi \) such that \( \xi(t_\lambda) = \psi \left[ A_d(t_\lambda) \right] \). Notice that \( \psi < 0 \), so \( \xi(t_\lambda) \) is decreasing in \( A_d(t_\lambda) \), and \( \psi(0) = \tilde{\xi} \). Intuitively, the larger the stock of inventories that dealers are holding at the time of the recovery, the lower the effective cost of holding the asset at the recovery time, and the longer it will take to deplete dealers’ inventories once the recovery has occurred.

\[23\]From (81), the reciprocal of \( \psi \) is defined as

\[
\psi^{-1}(\xi) = -\alpha \int_0^{\frac{\ln(\xi)}{\alpha}} e^{\alpha s} \left\{ A - \sum_i \pi_i U_i^{r-1} \left[ e^{(r+\kappa)s} \xi \right] \right\} ds
\]

60
Figure 9 shows the phase diagram of the dynamic system \([A_d(t), \xi(t)]\) following the recovery. From (22) we see that the \(A_d\)-isocline is upward-sloping and intersects the vertical axis at the steady-state point. The equilibrium trajectory of the economy is indicated in the figure by arrows along the saddle-path, namely, \(\xi(t) = \psi[A_d(t)]\). The initial condition \(A_d(t_\lambda)\) determines the starting point on the saddle path. The trajectories marked with dotted lines that do not follow the saddle path are solutions to the differential equations (22) and (38) but they either fail to satisfy the transversality condition or the requirement that the equilibrium path \(\xi(t)\) be continuous.

Next, we analyze the economy before the arrival of the recovery shock. Let \(V_i^\ell(a, t)\) denote the value function corresponding to an investor who has preference type \(i\) and is holding portfolio \(a\) at time \(t < T_\lambda\). Then, the investor’s value function satisfies

\[
V_i^\ell(a, t) = \tilde{V}_i^\ell(a) + \mathbb{E}_i \left\{ \mathbb{I}_{\{T_\lambda \leq \tilde{T}\}} e^{-r(\tilde{T} - t)} \max_{a'} \left[ V_{k(\tilde{T})}^h(a, \tilde{T}, T_\lambda) - p^h(\tilde{T}, T_\lambda) (a' - a) \right] \\
+ \mathbb{I}_{\{\tilde{T} < T_\lambda\}} e^{-r(\tilde{T} - t)} \max_{a'} \left[ V_{k(\tilde{T})}^\ell(a', \tilde{T}) - p^\ell(\tilde{T}) (a' - a) \right] \right\},
\]

(82)
where the indicator function \( I_{\{T_\lambda \leq \tilde{T}\}} \) equals one if \( T_\lambda \leq \tilde{T} \) and zero otherwise, and

\[
\tilde{V}_i^\ell(a) = \mathbb{E}_i \left\{ \int_t^{\tilde{T}} \left[ R + I_{\{s>T_\lambda\}}(1-R) \right] u_k(s) (a) e^{-r(s-t)} ds \right\}.
\]

This Bellman equation is a natural generalization of (9), for example, they coincide if we set \( R = 1 \) and let \( \lambda \to 0 \). The function \( \tilde{V}_i^\ell(a) \) is the expected discounted sum of utility flows that an investor enjoys from holding a quantity \( a \) of the asset until he gains effective access to the market at Poisson rate \( \kappa \). The term \( \left[ R + I_{\{s>T_\lambda\}}(1-R) \right] \) indicates that the investor’s instantaneous utility is scaled down by \( R \) until the economy recovers. It will be convenient to define \( U_i^\ell(a) = (r + \kappa)\tilde{V}_i^\ell(a) \). If \( \lambda = 0 \) then \( U_i^\ell(a) \) reduces to \( RU_i(a) \). Alternatively, as \( \lambda \to \infty \) (the economy recovers almost surely in the next instant), \( U_i^\ell(a) \to U_i(a) \).

The following Lemma gives a formulation of the investor’s problem which is analogous to the one in Lemma 2.

**Lemma 15** An investor of preference type \( i \) who holds portfolio \( a \) and gains direct effective access to the market at time \( t \) before the recovery has taken place, solves

\[
\max_{a_t^i} \left[ U_i^\ell(a^i_t) - \xi^\ell(t)a^i_t \right]
\]

where

\[
\xi^\ell(t) = (r + \kappa) \left[ p^\ell(t) - \int_0^{\infty} K e^{-(r+\kappa+\lambda)r^*} p^\ell(t + \tau\kappa) d\tau\kappa \right.
\]

\[
- \int_0^{\infty} \int_{\tau\kappa}^{\infty} \lambda e^{-\lambda\kappa} K e^{-(r+\kappa)r^*} p^h(t + \tau\kappa, t + \tau\kappa) d\tau\kappa d\tau\lambda \] .

**Proof.** The first term on the right-hand side of (82), \( \tilde{V}_i^\ell(a) \), satisfies the following flow Bellman equation,

\[
(r + \kappa)\tilde{V}_i^\ell(a) = Ru_i(a) + \delta \sum_j \pi_j \left[ \tilde{V}_j^\ell(a) - \tilde{V}_i^\ell(a) \right] + \lambda \left[ \tilde{V}_i(a) - \tilde{V}_i^\ell(a) \right],
\]

where \( \tilde{V}_i(a) = U_i(a)/(r + \kappa) \). The investor’s portfolio problem before the recovery corresponds to

\[
\max_{a} \left\{ \tilde{V}_i(a) - p^\ell(t)a - \mathbb{E} \left[ e^{-r(T-t)} \left[ I_{\{T<T_\lambda\}} p^\ell(T) + I_{\{T\geq T_\lambda\}} p^h(T, T_\lambda) \right] a \right] \right\}
\]

Using that \( U_i^\ell(a) = (r + \kappa)\tilde{V}_i^\ell(a) \), (86) can be reexpressed as

\[
\max_{a} U_i^\ell(a) - \xi^\ell(t)a
\]
where
\[
\xi^\ell(t) = [r + \kappa] \left\{ p^\ell(t) - \mathbb{E} \left[ \mathbb{I}_{\{T < T\}} e^{-r(T-t)} p^\ell(T) + \mathbb{I}_{\{T \leq T\}} p^h(T, T_\lambda) \right] \right\}
\]

Using the fact that \( T - t \) and \( T_\lambda - t \) are two independent random variables, exponentially distributed, the expected value of the resale price is
\[
\mathbb{E} \left[ e^{-r(T-t)} \left[ \mathbb{I}_{\{T < T\}} p^\ell(T) + \mathbb{I}_{\{T \geq T\}} p(T, T) \right] \right] = \\
\int_0^\infty \int_0^\infty e^{-rt} \left[ \mathbb{I}_{\{\tau < \tau_\lambda\}} p^\ell(t + \tau) + \mathbb{I}_{\{\tau \geq \tau_\lambda\}} p(t + \tau, t + \tau_\lambda) \right] \kappa e^{-\kappa \tau} \lambda e^{-\lambda \tau} d\tau d\tau_\lambda
\]

Changing the order of integration of the second term, one can derive (84).

According to Lemma 15, an investor maximizes his effective utility function, \( U^\ell_i(a) \), minus the effective cost of investing in the asset, \( \xi^\ell(t) a \). Just as \( U^\ell_i(a) \) takes into account both idiosyncratic and aggregate preference shocks, \( \xi^\ell(t) \) takes into account the expected capital gain that will be realized the next time the investor gains access to the market, which may be before or after the economy recovers. As before the last two terms on the right-hand side of (84) represent the expected resale price of the asset. From Lemma 15 it follows that during the crisis, an optimal portfolio choice \( a^\ell_i(t) \) satisfies
\[
U^\ell_i[a^\ell_i(t)] = \xi^\ell(t).
\] (87)

We now turn to analyze a dealer’s problem. At any time \( t \) before the recovery has occurred, the dealer solves
\[
\max_{q(s)} \mathbb{E} \left[ \int_t^{T_\lambda} e^{-r(s-t)} p^\ell(s) q(s) ds + e^{-r(T_\lambda-t)} \mathcal{W}^h [a_d(T_\lambda), T_\lambda, T_\lambda] \right],
\]
subject to \( a_d(s) = q(s), a_d(s) \geq 0 \) for all \( s \geq t \) and the initial condition \( a_d(t) \). Lemma 16 simplifies the dealer’s problem.

**Lemma 16** At any every time \( t \) before the recovery has occurred, the dealer solves
\[
\max_{a_d(t+s) \geq 0} \int_0^\infty e^{-(r + \lambda)s} \left\{ -rp^\ell(t + s) + p^\ell(t + s) + \lambda \left[ p^h(t + s, t + s) - p^\ell(t + s) \right] \right\} a_d(t + s) ds
\]
given an initial condition \( a_d(t) \).
Proof. Integration by parts and the fact that \( \lim_{t \to \infty} e^{-rt} p^h(s, t) a_d(t) = 0 \) (by Lemma 14) implies that (77) can be written as

\[
W^h(a_d, t, t) = W^h(0, t, t) + p^h(t, t) a_d,
\]

where \( W^h(0, t, t) = \max_{a_d(s) \geq 0} \int_t^\infty e^{-r(s-t)} [p^h(s, t) - rp^h(s, t)] a_d(s) ds. \) Integration by parts and (89) allow us to rewrite the dealer’s problem (88) as

\[
\max_{a_d(s) \geq 0} \mathbb{E} \left\{ \int_t^{T_\lambda} e^{-r(s-t)} \left[ \dot{p}^f(s) - rp^f(s) \right] a_d(s) ds + e^{-r(T_\lambda-t)} \left[ p^h(T_\lambda, T_\lambda) - p^f(T_\lambda) \right] a_d(T_\lambda) \right\}.
\]

After a change of variables, defining \( \tau_\lambda = T_\lambda - t \) and noticing that \( \tau_\lambda \) is an exponentially distributed random variable with mean \( 1/\lambda \), this last expression becomes

\[
\max_{a_d(t+s) \geq 0} \int_0^\infty \lambda e^{-\lambda \tau_\lambda} \int_0^{T_\lambda} e^{-rs} \left[ \dot{p}^f(t+s) - rp^f(t+s) \right] a_d(t+s) ds d\tau_\lambda + \int_0^\infty \lambda e^{-(r+\lambda) \tau_\lambda} \left[ p^h(t + \tau_\lambda, t + \tau_\lambda) - p^f(t + \tau_\lambda) \right] a_d(t + \tau_\lambda) d\tau_\lambda
\]

for \( s \geq t \), with \( a_d(t) \) given. Since (90) is the same as

\[
\max_{a_d(t+s) \geq 0} \int_0^\infty \int_0^\infty \lambda e^{-\lambda \tau_\lambda} e^{-rs} \left[ \dot{p}^f(t+s) - rp^f(t+s) \right] a_d(t+s) ds d\tau_\lambda + \int_0^\infty \lambda e^{-(r+\lambda) \tau_\lambda} \left[ p^h(t + \tau_\lambda, t + \tau_\lambda) - p^f(t + \tau_\lambda) \right] a_d(t + \tau_\lambda) d\tau_\lambda,
\]

we can change the order of integration in the first term and integrate with respect to \( \tau_\lambda \) to arrive at the dealer’s problem as formulated in the statement of the lemma. ■

From Lemma 16 we see that the flow of profit of dealers during the crisis has three components: the opportunity cost of holding the asset, \( rp^f(t+s) \), the capital gain while the economy remains in the crisis state, \( \dot{p}^f(t) \), and the expected capital gain \( p^h(t+s, t+s) - \dot{p}^f(t+s) \) if the economy recovers (which occurs with Poisson intensity \( \lambda \)). Clearly, \( \dot{p}^f(t) + \lambda [p^h(t, t) - \dot{p}^f(t)] > rp^f(t) \) is inconsistent with equilibrium (the dealer’s problem would have no solution). Let \( a_d^f(t) \) denote the solution to the dealer’s problem. The dealer’s necessary conditions are immediate from Lemma 16: as long as the economy is in the crisis state,

\[
\left\{ -rp^f(t) + \dot{p}^f(t) + \lambda [p^h(t, t) - \dot{p}^f(t)] \right\} a_d^f(t) = 0 \tag{91}
\]

for all \( t \), with \( a_d^f(t) \geq 0 \) and \( -rp^f(t) + \dot{p}^f(t) + \lambda [p^h(t, t) - \dot{p}^f(t)] \leq 0 \). The following lemma, which is analogous to Lemma 3 allows us express the dealer’s first-order conditions (91) in terms
of investors’ effective cost of buying the asset before the recovery and after the recovery has occurred. We use (78) to define

$$\xi^h (t, t\lambda) = \psi[A_d(t\lambda)] e^{-(r+\kappa)(t\lambda-t)}.$$  \hspace{1cm} (92)

Notice that given $\xi^h [t, t\lambda, A_d(t\lambda)]$ we can use (21) to find the path for the asset price after the recovery.\textsuperscript{24}

**Lemma 17** Condition (84) implies

$$-rp^f(t) + p^f(t) + \lambda \left[ p^h(t, t) - p^f(t) \right] = -\xi^f(t) + \frac{\dot{\xi}^f(t) + \lambda \left[ \xi^h(t, t) - \xi^f(t) \right]}{r+\kappa}.$$  \hspace{1cm} (93)

**Proof.** Let

$$P^f(t) = \int_t^\infty e^{-(r+\kappa+\lambda)(s-t)} p^f(s) \ ds$$

$$P^h(t) = e^{(r+\kappa+\lambda)t} \int_t^\infty \int_{s}^{\infty} \lambda e^{-\lambda z} e^{-(r+\kappa)s} p^h(s, z) \ dsdz,$$

which correspond to the second and third terms in (84), respectively, after a change of variables. Then (84) can be written more compactly as

$$\xi^f(t) = (r+\kappa) \left[ p^f(t) - P^f(t) - P^h(t) \right],$$  \hspace{1cm} (94)

and therefore,

$$\dot{\xi}^f(t) = (r+\kappa) \left[ \dot{p}^f(t) - \dot{P}^f(t) - \dot{P}^h(t) \right].$$  \hspace{1cm} (95)

Note that

$$\dot{P}^f(t) = (r+\kappa+\lambda) P^f(t) - \kappa p^f(t)$$

and

$$\dot{P}^h(t) = (r+\kappa+\lambda) P^h(t) - \lambda \int_t^\infty e^{-(r+\kappa)(s-t)} p^h(s, t) \ ds.$$  \hspace{1cm} (96)

From the investor’s problem ((12) and Lemma 3), we know that

$$\xi^h(t, t\lambda) = (r+\kappa) \left[ p^h(t, t\lambda) - \int_t^\infty e^{-(r+\kappa)(s-t)} p^h(s, t\lambda) \ ds \right],$$

\textsuperscript{24}Specifically, $p^h(t, t\lambda) = \int_t^\infty e^{-(r+\kappa)(s-t)} \left[ \xi^h(s, t\lambda) - \frac{\xi^h(s, t\lambda)}{r+\kappa} \right] \ ds$, where hereafter, $\xi^h(s, t\lambda)$ is used to denote $\partial \xi^h(s, t\lambda) / \partial s$ and $p^h(t, t\lambda)$ to denote $\partial p^h(t, t\lambda) / \partial t$. \hspace{1cm} (97)
which evaluated at \( t_\lambda = t \) implies

\[
\int_t^\infty \kappa e^{-(r+\kappa)(s-t)} p^h(s,t)ds = p^h(t,t) - \frac{\xi^h(t,t)}{r+\kappa}.
\]  

Substituting (97) back into (96) we get

\[
\dot{P}^h(t) = (r + \kappa + \lambda) P^h(t) - \lambda \left[ p^h(t,t) - \frac{\xi^h(t,t)}{r+\kappa} \right].
\]  

Next, substitute (95) and (98) into (94) to arrive at

\[
\frac{\xi^f(t)}{r+\kappa} = \dot{p}^f(t) - (r + \kappa + \lambda) \left[ P^f(t) + P^h(t) \right] + \kappa p^f(t) + \lambda \left[ p^h(t,t) - \frac{\xi^h(t,t)}{r+\kappa} \right],
\]

which after using (93) to substitute \( [P^f(t) + P^h(t)] \) and rearranging reduces to

\[-rp^f(t) + \dot{p}^f(t) + \lambda \left[ p^h(t,t) - p^f(t) \right] = -\xi^f(t) + \frac{\dot{\xi}^f(t) + \lambda \left[ \xi^h(t) - \xi^f(t) \right]}{r+\kappa},\]

the expression in the statement of the lemma.  

Lemma 17 allows us to write (91) as

\[
\left\{ \dot{\xi}^f(t) + \lambda \xi^h(t,t) - (r + \kappa + \lambda) \xi^f(t) \right\} a^f_d(t) = 0.
\]

To summarize, we have shown that once the economy has recovered from the crisis, say at some time \( t_\lambda \), it will evolve along a deterministic path \( \{ A^h_d(t), \xi^h(t,t_\lambda) \} \) given by (79) and (92). Before it has recovered from the crisis, it follows a path \( \{ A^f_d(t), \xi^f(t) \} \) which, using (92), satisfies

\[
\left\{ \dot{\xi}^f(t) + \lambda \psi[A^f_d(t)] - (r + \kappa + \lambda) \xi^f(t) \right\} A^f_d(t) = 0
\]

and the market clearing condition

\[
\dot{A}^f_d(t) = \alpha \left\{ A - A^f_d(t) - \sum_i \pi_i U_{it}^{\theta - 1}[\xi^f(t)] \right\}.
\]  

We can now define an equilibrium to be a stochastic process \( \{ \xi(t), A_d(t) \} \), such that for \( t < T_\lambda \), \( \{ \xi(t), A_d(t) \} = \{ \xi^f(t), A^f_d(t) \} \) satisfying (99) and (100), and for \( t \geq T_\lambda \), \( \{ \xi(t), A_d(t) \} = \{ \xi^h(t,T_\lambda), A^h_d(t) \} \) satisfying (79) and (92).

Let \( (\xi^f, \bar{A}^f_d) \) denote the steady-state associated with (99) and (100); it is characterized by

\[
\begin{align*}
\dot{\xi}^f &\geq \frac{\lambda}{r + \kappa + \lambda} \psi(\bar{A}^f_d) \quad \text{" if } \bar{A}^f_d > 0 \\
A &= \bar{A}^f_d + \sum_i \pi_i U_{it}^{\theta - 1}(\xi^f)
\end{align*}
\]

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As the random time of recovery, $T\lambda$, becomes very large, $\{\xi_{d}(t), A_{d}^{\ell}(t)\}$ approach their steady state values $(\bar{\xi}, \bar{A}_{d}^{\ell})$ as given by (101) and (102). Assuming $\bar{A}_{d}^{\ell} > 0$ it can be checked from (99) and (100) that the steady state is a saddle point and that there is a unique trajectory that brings the system to its steady state.

**Proof of Proposition 4** Dealers accumulate inventories if and only if $\bar{A}_{d}^{\ell} > 0$. From (101) and (102), $\bar{A}_{d}^{\ell}$ is determined by the condition $\Gamma(\bar{A}_{d}^{\ell}) = 0$, where

$$
\Gamma(\bar{A}_{d}) = A_{d} + \sum_{i} \pi_{i} U_{i}^{r-1} \left[ \frac{\lambda}{r + \kappa + \lambda \psi(A_{d})} \right] - A.
$$

Since $\Gamma'(A_{d}) > 0$ and $\lim_{A_{d} \to \infty} \Gamma(A_{d}) = \infty$, there is a unique $\bar{A}_{d}^{\ell} > 0$ such that $\Gamma(\bar{A}_{d}^{\ell}) = 0$ iff $\Gamma(0) < 0$. Using the fact that $\psi(0) = \bar{\xi}$ we know that $\Gamma(0) = \sum_{i} \pi_{i} U_{i}^{r-1} \left( \frac{\lambda}{r + \kappa + \lambda \bar{\xi}} \right) - A$, so $\Gamma(0) < 0$ is equivalent to (41).

**Derivation of (43).** From (81), normalizing $t_{\lambda}$ to 0 and assuming that $A_{d}(t_{\lambda}) = \bar{A}_{d}^{\ell}$, we get

$$
\bar{A}_{d}^{\ell} + A \int_{0}^{T} e^{\alpha s} \left[ A - \sum_{i} \pi_{i} U_{i}^{r-1} \left[ \xi(s) \right] \right] ds = 0
$$

Assuming the functional form $u_{i}(\alpha) = \varepsilon_{i} \alpha^{1-\sigma}/(1 - \sigma)$ we have $U_{i}^{r-1} \left[ \xi(s) \right] = \left[ \varepsilon_{i}/\xi(s) \right]^{1/\sigma}$. Furthermore, $\xi(s) = \bar{\xi} e^{(r + \kappa)(T - s)}$. Hence,

$$
\bar{A}_{d}^{\ell} + A \int_{0}^{T} e^{\alpha s} \left[ A - \sum_{i} \pi_{i} \left[ \frac{\varepsilon_{i}}{\bar{\xi}} \right]^{1/\sigma} e^{(r + \kappa)(T - s)} \right] ds = 0
$$

Notice that $\sum_{i} \pi_{i} \left[ \frac{\varepsilon_{i}}{\bar{\xi}} \right]^{1/\sigma} = A$. Thus, after some calculation, we have

$$
\frac{\bar{A}_{d}^{\ell} - A}{A} + \frac{\gamma}{\gamma - \alpha} e^{\alpha T} - \frac{\alpha}{\gamma - \alpha} e^{\gamma T} = 0,
$$

(103)

where $\gamma \equiv \frac{r + \kappa}{\sigma}$. The steady-state condition (102) yields

$$
A = \bar{A}_{d}^{\ell} + \sum_{i} \pi_{i} \left[ \frac{\varepsilon_{i}}{\bar{\xi}} \right]^{1/\sigma}
$$

Combined with (101), $\bar{\xi} = \frac{\lambda}{r + \kappa + \lambda \bar{\xi}} e^{-(r + \kappa)T}$, it implies

$$
A = \bar{A}_{d}^{\ell} + e^{\gamma T} \sum_{i} \pi_{i} \left[ \frac{r + \kappa + \lambda \bar{\xi}}{\bar{\xi}} \right]^{1/\sigma}
$$

(104)

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Using the fact that $\xi^1 = \sum_i \pi_i [\xi_i]^{1/\sigma} / A$, (104) becomes

$$e^{\gamma T} \Omega = \frac{A - A_d}{A}$$

(105)

where $\Omega \equiv \left( \frac{r + \kappa + \lambda}{\lambda} \right)^{1/\sigma} \sum_i \pi_i (\bar{\xi}_i)^{1/\sigma} / \sum_i \pi_i (\bar{\xi}_i)^{1/\sigma}$. Plug (105) into (103) to obtain

$$T = \frac{1}{\alpha - \gamma} \ln \left[ \frac{\alpha}{\gamma} + \left( \frac{\gamma - \alpha}{\gamma} \right) \Omega \right]$$

(106)

Substitute the expression for $T$ given by (106) into (105) to obtain (43).