PERSISTENCE ANALYSIS OF HEDGE FUND RETURNS

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Abstract

We use a model of Markov chain to evaluate the pure persistence in hedge fund returns. We study two forms of pure persistence: absolute persistence (positive/negative returns) and persistence with respect to a high water mark (accounting for the amplitude of drawdowns). In the first case, we find that hedge funds in general exhibit persistence of positive returns, but no persistence in negative ones. In contrast, the results using the high water mark criterion show the presence of both positive and negative persistence. In order to account for the presence of serial correlation, we use a new approach based on the method of moments and on the model of Getzmannsky et al. (2004). Our approach avoids imposing a specific MA model for the unsmoothing process allowing for more accurate results. Our findings suggest that the smoothing contributes to an increase in absolute persistence. These results suggest also that hedge fund managers exhibit a relatively high probability to deliver positive returns, but a much weaker probability to increase their high water mark.

JEL Classification: C13, G11, G23.

Keywords: Hedge funds, Markov chain, smoothed returns, persistence, high water mark.

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1. Introduction

The last few years have provided a challenging environment for hedge fund managers. As the number of hedge funds approaches the 10,000 milestone and assets under management have already surpassed the two trillion dollar mark, it is only natural that investor become increasingly skeptical about the ability of the hedge fund industry to continue to offer significant value. The absolute returns that have long been advertised by hedge fund managers have been increasingly hard to come by over the last few years, and it is estimated that approximately 80% of hedge funds have been in the red over the year 2008. The increased market volatility, the subprime debacle and ensuing credit crunch have recently added to an already difficult investment environment, however, given the exorbitant fee structure of these funds, investors have come to expect strong performance regardless of market conditions. The performance of these funds has been scrutinized by both practitioners and academics, and hedge fund managers are increasingly suspected of selling beta returns (returns linked to readily available market risk premia) as opposed to alpha (absolute) returns. Given the changing nature of the hedge fund universe, it is therefore vital to identify the managers who can systematically provide positive returns, also referred to as pure persistence.

In the area of persistence evaluation, a distinction has to be made between relative persistence and the pure persistence. In evaluating relative persistence, funds of the same strategy are classified as winners or losers depending on their performance relative to the median return over a given period. Evidence of persistence is found when winners and/or losers maintain their classification in two subsequent periods. Most of the studies in hedge fund literature address the question of persistence in terms of relative persistence and adopt many of the tests employed in mutual fund literature where this notion has been widely addressed. Relative persistence studies provide a general picture of whether past performance is a reliable indicator of future performance within a peer-group comparison framework. It doesn’t isolate a specific fund and analyze its performance over time; this is achieved by investigating pure persistence. Pure persistence aims to identify funds that systematically generate positive returns. Even though the study of pure persistence could be informative in the mutual fund context, it doesn’t have the same relevance as relative persistence given that mutual fund managers are index trackers and they are evaluated relative to their benchmark. Losses incurred by mutual fund managers are not necessarily classified as bad as long as the managers outperform their benchmark; the fact that managers are not evaluated relative to an exogenous threshold explains why there is no significant literature on pure persistence in mutual fund performance. Nonetheless, even if the studies on persistence analysis in hedge fund performance followed the same trend, it is important to note that the evaluation of the managers is not done in the same manner. Hedge funds are absolute returns strategies and investors expect absolute returns (good returns) regardless of the direction of the market. High incentive fees charged by hedge fund managers (which average 20%) are then supposed to justify this privilege and the latter are not evaluated relative to a benchmark, but on their ability to deliver absolute returns. The fact that recent studies (among which Hasanhodzic and Lo (2007)) show that a larger proportion of hedge funds are exposed to beta driven returns calls into question the high level of incentives fees charged to investors. In the case of hedge funds, the analysis of pure persistence provides a more appropriate measure than relative persistence analysis, and allows us to identify managers exhibiting
superior skills in terms of absolute performance; and in the current context of financial crisis where investors are increasingly aware of the fact that finding a manager able to deliver absolute returns became a challenge, the pure persistence analysis becomes more than ever relevant.

As mentioned above, the majority of studies investigate the relative persistence in hedge fund returns. Brown, Goetzmann and Ibbotson (1999), Agarwal and Naik (2000) and Liang (2000) use parametric tests (cross-sectional regressions) and non-parametric tests (Cross Product Ratio, Chi-square test, Kolomogorov-Smirnov test) to investigate the presence of relative persistence in hedge fund returns. They find no evidence of relative persistence at annual horizons even if Agarwal and Naik (2000) find that hedge fund returns persist in the short term. More recently, Kosowski, Naik and Teo (2007) use a Bayesian approach to improve the accuracy of alpha estimates. They find evidence of long term relative persistence and argue that one reason why the previous studies do not find the same results is that they rely on relatively imprecise performance measures. As for pure persistence, De Souza and Gokcan (2004) use the Hurst exponent, combined with a D-statistic to study a relatively small sample of funds. They find that the funds exhibiting the strongest persistence of positive returns during the in-sample period (36 months) showed a better risk-adjusted profile in the out-of sample period. However, the accuracy of the results remains a problem in their evaluation because one of the disadvantages with the Hurst exponent is that it requires a large sample to get significant results.

In this paper, we address the performance of hedge funds in terms of pure persistence. The contribution of our study is threefold. Firstly, we evaluate the pure persistence of hedge funds with a new approach using a model of Markov chains. The persistence is then evaluated in terms of transition probabilities. The probabilities have the advantage of not supposing an a priori distribution of returns and are easily interpretable. Moreover, for our analysis we define two kind of persistence: the absolute persistence (positive/negative returns) and the persistence with respect to a high water mark. It is well known that several hedge fund strategies, in particular arbitrage strategies carry out positive returns of small amplitude in general, but when they face losses, they are often of larger amplitudes; the analysis of absolute persistence does not capture this aspect because it does not take into account the amplitude positive or negative returns, it only focuses on the sign of returns. It follows that two managers exhibiting the same sequence of positive and negative returns on a given period would obtain the same evaluation in terms of absolute performance, regardless of the fact that one might have incurred substantially greater losses. One way to address this issue is to take into account the size of returns and to evaluate the persistence with respect to a high water mark. The high water mark represents the greatest value reached by an investment during a period. A manager who carries out small positive returns in general but faces large losses during the investment period will have trouble surpassing his high water mark. It could take considerable time for certain managers to reach their high water mark after a significant drawdown. The analysis of persistence with respect to a high water mark will then consist to assess the ability to increase the high water mark in a sustainable way.

Secondly, we develop a method to test the significance of persistence estimates regardless of the length of the sample. This helps to avoid the problem that one could face by using the Hurst exponent in small samples. For this purpose, we use a one tailed t-test which allows seeing whether a transition probability is statistically superior to 0.5.
Finally, we evaluate the persistence before and after taking into account the serial correlation in hedge fund returns. Several studies (Asness, Krail and Liew (2001), Brooks and Kat (2002), Okunev and White (2003), Getmansky, Lo and Makarov (2004)) identify the presence of significant serial correlation in hedge fund returns, which basically leads to an underestimation of their real risk. Getmansky et al. (2004) argue that the most likely source of serial correlation in hedge fund returns is the smoothing of returns due to illiquidity and to managers’ personal motivation to optimize their performance over several periods. Illiquidity because many hedge strategies invest in illiquid assets like non-quoted assets in private equity, some emerging market stocks and bonds, real estate and infrastructure, etc. In the event that managers smooth the reported returns, the disclosed volatility will be smaller than the realised volatility and hence would upwardly bias measure of pure persistence. Getmansky et al. (2004) propose an econometric model based on an MA(2) approach to unsmooth returns. Okunev and White (2003) use a method developed by Geltner (1993) in order to obtain a new corrected series. In this study, we use a model based on the method of moments to unsmooth returns. The advantage of our model is that it allows seeing when it is possible to have satisfactory solutions (positive weights) when one tries to unsmooth returns. Indeed, hedge fund returns don’t have the same order of serial correlation, and imposing an order of serial correlation for all funds like in Getmansky et al. (2004) could lead to unsatisfactory results. In their paper, they obtain negative coefficients for some funds whereas theoretically and according to the assumption of their model, all the weights should be positive. They argue that this can be attributed to a mis-specification of the model and that a different un-smoothing model may be more appropriate. In addition, contrary to the model of Getmansky et al. (2004), our model doesn’t assume normality for the estimation of the weighting coefficients.

The rest of the paper is organized as follows. Section 2 describes the methodology used to test the significance of the transition probabilities and in section 3, we present the methodology used to unsmooth returns. Section 4 presents the data and section 5 shows the results of the analysis. We conclude the study in section 6.

2. Methodology to measure pure persistence

Contrary to De Souza and Gokcan (2004), pure persistence will be firstly evaluated here in terms of probability of positive or negative returns over two periods. There are many advantages of using the probabilities in the performance evaluation. They do not make any assumption as to the distribution of returns and they are more easily interpretable for an investor than the combined analysis of the Hurst exponent and the D-statistic. Moreover, probabilities allow for an approximation of odds that a fund gets desirable returns, which is not the case for other measures like the mean of returns. Indeed, the mean gives the average performance of a manager over a period, but doesn’t indicate how the manager performs on a regular basis. For example, an average of 2% indicates that on aggregate, the manager’s performance is over zero, but does not indicate at which frequency he obtained positive returns or what the odds are that he provides positive returns. For instance, a fund could exhibit the following returns: -2%, -1%, 15%, -1.2%, -0.8%. This gives a mean of 2%, which is higher than 0%, but the fund has 4 odds out of 5 or a probability of 80% of having negative returns.

Another advantage of using probabilities in the relation between past and future returns is that contrary to serial correlation which is only relevant for elliptical distributions and which measures the
linear dependence between the returns, probabilities apply for other distributions and can measure dependence which might be non-linear. And we know from the literature that hedge fund returns are often non gaussian, due to the use of derivatives and dynamic strategies (Fung and Hsieh (1997), Agarwal and Naik (2004), etc.).

The evaluation of persistence is done through a model of Markov chains. The persistence is then measured in terms of transition probabilities. A Markov chain is a stochastic process where the prediction of the future depends on the present and is independent of the past. The set of possible values that the random variable can take is referred to as the state space and the markovian property is defined as follows:

\[ \Pr[X_{t+1} = j|X_0 = i_0, \ldots, X_{t-1} = i_{t-1}, X_t = i] = \Pr[X_{t+1} = j|X_t = i] \]  

(1)

where \( t \) represents the time for the states \( i_0, \ldots, i_{t-1}, i, j \). We will use a two state Markov chain to evaluate the persistence. Let \( R_t \), denotes the return of the fund at time \( t \) and \( I_t \) a dichotomus variable which follows the process:

\[
I_t = \begin{cases} 
1 & \text{if } R_t > 0 \\
0 & \text{if } R_t \leq 0
\end{cases}
\]

(2)

The series derived from this transformation follows a two state Markov chain and identifies strictly positive returns as 1 and negative or null returns as 0. The corresponding transition matrix is:

\[
M = \begin{bmatrix} p_{11} & p_{10} \\ p_{01} & p_{00} \end{bmatrix}
\]

with

\[
\begin{align*}
p_{11} &= \Pr[I_{t+1} = 1|I_t = 1] \\
p_{10} &= \Pr[I_{t+1} = 0|I_t = 1] \\
p_{01} &= \Pr[I_{t+1} = 1|I_t = 0] \\
p_{00} &= \Pr[I_{t+1} = 0|I_t = 0]
\end{align*}
\]

The elements in the diagonal of the transition matrix \( (p_{11} \text{ and } p_{00}) \) identify the presence of positive and negative persistence of returns. The probability \( p_{01} \text{ and } p_{10} \) indicate the probabilities of obtaining a gain after a lost, and vice versa. The transition probabilities are calculated to maximise the following likelihood function:

\[
L(S_T, \ p, \ \pi) = \log \pi + \sum_{ij=00}^{11} N_{ij} \log p_{ij} + M_{ij} \log(1 - p_{ij})
\]

(3)

4
where $S_T$ is the set of realized $I_t$, and $\pi$ the probability of the initial state. The latter can take the following values:

- If the initial state $I_1 = 1$
  \[ \pi = \pi_1 = \frac{1 - p_{00}}{2 - p_{11} - p_{00}} \]  \hspace{1cm} (4)

- If the initial state $I_1 = 0$
  \[ \pi = \pi_0 = \frac{1 - p_{11}}{2 - p_{11} - p_{00}} \]  \hspace{1cm} (5)

$N_{ij}$ and $M_{ij}$ are the occurrences associated with the various transitions. It is important to notice that $\pi$ is function of the transition probabilities\(^1\).

In this context of persistence analysis of hedge fund returns with limited historical data, it is important to ensure the significance of the transition probabilities. For this purpose, we developed an approach to test whether persistence estimators are statistically significant. To our knowledge, the existing tests in the literature for Markov chains consist most on independence or random walk tests and are generally based on likelihood ratio tests or $\chi^2$-tests\(^2\). For example, we know that $p_{11} > 0.5$ indicates positive persistence and $p_{00} > 0.5$ indicates negative persistence. Therefore, testing for positive persistence is equivalent to performing the following unilateral test:

\[
H_0 : p_{11} \leq 0.5 \\
H_1 : p_{11} > 0.5
\]

The corresponding t-statistic is:

\[ t = \frac{\hat{p}_{11} - 0.5}{\hat{\sigma}_{p_{11}}} : t_c(n - 1) \]  \hspace{1cm} (6)

Hence, we require the volatility estimate $\hat{\sigma}_{p_{11}}$. To this end, we estimate firstly the asymptotic value of $\text{Var} \left[ \sqrt{n} (\hat{p}_{11} - p_{11}) \right]$ where $p_{11}$ is the asymptotic value of the transition probability. This is achieved via the Delta method described below. We know that $\hat{p}_{11}$ can also be expressed in the following way:

\[ \hat{p}_{11} = \frac{\hat{P}_{11}}{\hat{P}_{11} + \hat{P}_{10}} \]  \hspace{1cm} (7)

where $\hat{P}_{11} = \Pr(I_t = 1; I_{t+1} = 1)$ and $\hat{P}_{10} = \Pr(I_t = 1; I_{t+1} = 0)$ are jointed probabilities. Thus, $\hat{p}_{11}$ is a function of $\hat{P}_{11}$ and $\hat{P}_{10}$ and we can write:

\(^1\)For more informations, the reader can refer to *Time Series Analysis*, J. D. Hamilton, Princeton University, 1994.

\[ \hat{p}_{11} = f(\hat{P}_{11}, \hat{P}_{10}) \]

By the Delta method and with some assumptions, we can show that:

\[ V ar(\sqrt{n}(\hat{p}_{11} - p_{11})) = V ar\left(\sqrt{n}(\hat{P}_{11})\right) + V ar\left(\sqrt{n}(\hat{P}_{10})\right) - 2Cov\left(\sqrt{n}(\hat{P}_{11}), \sqrt{n}(\hat{P}_{10})\right) \] (8)

In the appendix, we show that when \( n \to \infty \):

\[
\begin{align*}
V ar\left(\sqrt{n}\hat{P}_{11}\right) & \to \frac{5}{16} \\
V ar\left(\sqrt{n}\hat{P}_{10}\right) & \to \frac{1}{16} \\
Cov\left(\sqrt{n}\hat{P}_{11}, \sqrt{n}\hat{P}_{10}\right) & \to -\frac{1}{16}
\end{align*}
\]

This gives

\[ V ar(\sqrt{n}(\hat{p}_{11} - p_{11})) = \frac{1}{2} \]

From this result and the central limit theorem, it obtains that:

\[ \sqrt{n}(\hat{p}_{11} - p_{11}) \to N(0, \frac{1}{2}) \]

Therefore

\[ \hat{\sigma}_{p_{11}} = \frac{\sqrt{1/2}}{\sqrt{n}} = \frac{1}{\sqrt{2n}} \] (9)

We follow the same procedure for \( p_{00} \) (in appendix) and the results show that

\[ \sqrt{n}(\hat{p}_{00} - p_{00}) \to N(0, \frac{1}{2}) \]

and

\[ \hat{\sigma}_{p_{00}} = \frac{\sqrt{1/2}}{\sqrt{n}} = \frac{1}{\sqrt{2n}} \] (10)

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3 The demonstration can be found in appendix A.
4 We made a bootstrapping with a large sample of data and the variance converges towards 1/2.
3. Methodology to unsmooth the returns

In this study, we estimate the persistence for smoothed and unsmoothed returns for each fund. This enables us to verify whether the smoothing of returns has an effect on the persistence and if so, which strategies are the most affected. Getmansky, Lo and Makarov (2004) (henceforth GLM) propose a model using maximum likelihood estimation to obtain for the "unsmoothed" time series of returns. The model of GLM assumes that the observed return in period \( t \) \( (R_t^o) \) is a weighted average of the "true" return \( (R_t^c) \) over the most recent \( k + 1 \) periods, including the current period:

\[
R_t^o = \theta_0 R_t^c + \theta_1 R_{t-1}^c + \cdots + \theta_k R_{t-k}^c
\]  
(11)

\[
\theta_j \in [0, 1], \quad j = 0, ..., k
\]  
(12)

\[
1 = \theta_0 + \theta_1 + \cdots + \theta_k
\]  
(13)

The \( \theta_s \) can be estimated using the maximum likelihood approach. The smoothing level (or smoothing index) is equal to the sum of the squared \( \theta_j \):

\[
\xi = \sum_{j=0}^{k} \theta_j^2
\]  
(14)

By construction \( 0 \leq \xi \leq 1 \). A small value of \( \xi \) implies a high smoothing level, \( \xi = 1 \) indicates no smoothing. After estimating the \( \theta_s \), the "true" returns (unsmoothed) are obtained by inverting the equation in this way:

\[
R_t^c = \frac{R_t^o - \hat{\theta}_1 R_{t-1}^c - \cdots - \hat{\theta}_k R_{t-k}^c}{\hat{\theta}_0}
\]  
(15)

The unsmoothed and the observed returns have the same mean, but not the same variance. The variance of the unsmoothed returns is higher than that of the observed returns \( (\sigma_c^2 \geq \sigma_o^2) \) and the variance of the observed returns is \( \xi \) times smaller than that of the unsmoothed ones : \( \sigma_o^2 = \xi \sigma_c^2 \).

To estimate the \( \theta_s \), GLM first centered the observed returns to come up with a new time series

\[
X_t = R_t^o - \mu
\]  
(16)

Given the process described before the equation becomes:

\[
X_t = R_t^o - \mu = \theta_0(R_t^c - \mu) + \theta_1(R_{t-1}^c - \mu) + \cdots + \theta_k(R_{t-k}^c - \mu) + (\theta_0 + \theta_1 + \cdots + \theta_k)\mu - \mu
\]  
(17)

Setting \( R_t^c - \mu = \eta_t \), \( R_{t-1}^c - \mu = \eta_{t-1} \), ..., \( R_{t-k}^c - \mu = \eta_{t-k} \), we get :

\[
X_t = \theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k}
\]  
(18)
where the last assumption is added for purposes of estimation of the $MA(k)$ process.

In their model, GLM estimate the $\theta$s for 909 hedge funds with a $MA(2)$ assuming a serial correlation of lag 2 for hedge fund returns. This method is very attractive but nevertheless raises some problems. On the one hand, it is based on the assumption that demeaned ($\eta_t$) returns follow a normal distribution and the authors mention that although the maximum likelihood estimation has some attractive properties it is consistent and asymptotically efficient under certain regularity conditions. Therefore, it may not perform well in small samples or when the underlying distribution of true returns is not normal. Moreover, GLM mention that even if the normality condition is satisfied and a sufficient sample size is available, the smoothing model may simply not apply to some funds. If the numerical optimization does not converge it could be due to the fact that the model is misspecified, either because of non-normality or because of an inappropriate model’s specification. Another check is to verify whether or not the estimated smoothing coefficients are all positive in sign. Estimated coefficients that are negative and significant may be a sign that the constraint of positivity (of weights) is violated, which suggests that a somewhat different smoothing model may apply. In their study which imposes an $MA(2)$ specification, they obtain negative weights (negative values for $\theta_1$ and $\theta_2$) for some funds. It is important to note that all the funds don’t have the same level of serial correlation, therefore imposing the same level of serial correlation for all funds could lead to estimation of misspecified parameters $\theta_j$ and this could have undesirable effects on the distribution of unsmoothed returns. For example, when a parameter $\theta_j$ is negative, the fact that the weights must sum to 1 implies that at least one of them could be superior to 1. In that case we will have a smoothing level $\xi > 1$ and the variance of unsmoothed returns will be lower than the variance of the observed returns, which will underestimate the true risk of the fund. This suggests that it is very important to specify the appropriate model for each fund. For example, funds investing in liquid securities will probably have serially uncorrelated returns and imposing the unsmoothing of their returns could lead to misspecified $\theta$s. This is why it is first of all important to check the level of the serial correlation of returns.

In this study, we propose a model based on the method of moments to estimate the $\theta$s. Our model has the advantage of identifying when it is possible to have a satisfactory solution for $\theta$s. In addition, our model doesn’t assume normality; this is a relevant point given that many studies documented the non-normality of hedge fund returns. Let us reconsider the model of GLM (2004):

\begin{align}
1 &= \theta_0 + \theta_1 \eta_{t-1} + \ldots + \theta_k \eta_{t-k} \\
1 &= \theta_0 + \theta_1 \ldots + \theta_k \\
\eta_t &\sim D(0, \sigma^2_{\eta})
\end{align}

where in this case the demeaned $\eta_t$ follows a distribution $D$ which is not necessarily normal. We only suppose that the unobserved returns are independent and have a constant volatility to estimate.
Suppose the observed returns are serially correlated up to lag \( k \). By using the method of moments, it implies:

\[
E [X_t^2] = E [(\theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k})(\theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k})]
\]

\[
= \theta_0^2 \sigma^2_{\eta} + \theta_1^2 \sigma^2_{\eta} + \cdots + \theta_k^2 \sigma^2_{\eta}
\]

\[
= (\theta_0^2 + \theta_1^2 + \cdots + \theta_k^2) \sigma^2_{\eta}
\]

\[
E [X_t, X_{t-1}] = E [(\theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k})(\theta_0 \eta_{t-1} + \theta_1 \eta_{t-2} + \cdots + \theta_k \eta_{t-k-1})]
\]

\[
= \theta_0 \theta_1 \sigma^2_{\eta} + \theta_0 \theta_2 \sigma^2_{\eta} + \cdots + \theta_{k-1} \theta_k \sigma^2_{\eta}
\]

\[
= (\theta_0 \theta_1 + \theta_0 \theta_2 + \cdots + \theta_{k-1} \theta_k) \sigma^2_{\eta}
\]

\[
E [X_t, X_{t-2}] = E [(\theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k})(\theta_0 \eta_{t-2} + \theta_1 \eta_{t-3} + \cdots + \theta_k \eta_{t-k-2})]
\]

\[
= \theta_0 \theta_2 \sigma^2_{\eta} + \theta_0 \theta_3 \sigma^2_{\eta} + \cdots + \theta_{k-2} \theta_k \sigma^2_{\eta}
\]

\[
= (\theta_0 \theta_2 + \theta_0 \theta_3 + \cdots + \theta_{k-2} \theta_k) \sigma^2_{\eta}
\]

\[
E [X_t, X_{t-k}] = E [(\theta_0 \eta_t + \theta_1 \eta_{t-1} + \cdots + \theta_k \eta_{t-k})(\theta_0 \eta_{t-k} + \theta_1 \eta_{t-k-1} + \cdots + \theta_k \eta_{t-2k})]
\]

\[
= \theta_0 \theta_k \sigma^2_{\eta}
\]

Thus, we have \( k \) moment conditions, and we want to estimate \( k + 1 \) parameters. We also have one more condition which is \( \sum_j \theta_j = 1 \). This leads to a system of \( k + 1 \) equations with \( k + 1 \) unknown parameters:

\[
\left\{ \begin{array}{l}
E [X_t^2] = (\theta_0^2 + \theta_1^2 + \cdots + \theta_k^2) \sigma^2_{\eta} \\
E [X_t, X_{t-1}] = (\theta_0 \theta_1 + \theta_0 \theta_2 + \cdots + \theta_{k-1} \theta_k) \sigma^2_{\eta} \\
E [X_t, X_{t-2}] = (\theta_0 \theta_2 + \theta_0 \theta_3 + \cdots + \theta_{k-2} \theta_k) \sigma^2_{\eta} \\
\vdots \\
E [X_t, X_{t-k}] = \theta_0 \theta_k \sigma^2_{\eta} \\
1 = \theta_0 + \theta_1 + \cdots + \theta_k
\end{array} \right. \]  

(28)

We are then able to estimate the parameters. One way to do that with simplicity is to first estimate the order \( k \) of serial correlation of the observed returns. In the GLM model, they assume that all the funds have returns serially correlated up to lag 2, what is not necessarily true. For example, Managed futures funds may have for most of them uncorrelated returns or returns correlated up to lag 1, because they invest generally in liquid securities, and imposing a level of serial correlation could lead to misspecified parameters. Our approach is to first measure the level of serial correlation and estimate after the corresponding parameters \( \theta_j \) and \( \sigma^2_{\eta} \). We will limit the development up to lag 2. Depending on the level of serial correlation found, we have three main cases:
a) First case : \( k = 0 \)

If the first and the second order of serial correlation are not statistically significant, it is not necessary to unsmooth the returns and we keep them as they are.

b) Second case : \( k = 1 \)

If the first order of serial correlation is statistically significant and not the second one, we have 3 parameters to estimate \( \theta_0, \theta_1 \) and \( \sigma^2_\eta \) from the following system of equations :

\[
\begin{align*}
E \left[ X_t^2 \right] &= (\theta_0^2 + \theta_1^2) \sigma^2_\eta \\
E \left[ X_t X_{t-1} \right] &= \theta_0 \theta_1 \sigma^2_\eta \\
1 &= \theta_0 + \theta_1
\end{align*}
\]

The resolution of this system of equations gives the following results\(^5\):

\[
\begin{align*}
\sigma^2_\eta &= E \left[ X_t^2 \right] + 2E \left[ X_t X_{t-1} \right] \\
\theta_0 &= \frac{1}{2} \frac{\sqrt{1 - 4\gamma_1}}{2} \\
\theta_1 &= \frac{1}{2} \frac{\sqrt{1 - 4\gamma_1}}{2}
\end{align*}
\]

with

\[
\gamma_1 = \frac{E \left[ X_t X_{t-1} \right]}{\sigma^2_\eta}
\]

Then, the solutions of the system exist if and only if \( \gamma_1 \leq \frac{1}{4} \) and to have satisfactory solutions \((\theta_1 \geq 0)\), \( \gamma_1 \) should lead in this interval:

\[
0 \leq \gamma_1 \leq \frac{1}{4}
\]

The first order of autocorrelation should neither too high nor negative because if \( \gamma_1 < 0 \) i.e. if \( \text{Cov}(X_t, X_{t-1}) < 0 \), we will have \( \theta_1 < 0 \). In other words, if the serial correlation of order 1 is negative, all the weights won’t be positive and the unsmoothing will be incongruous because \( \xi \) will be higher than 1 and \( \sigma^2_\eta \) will be lower than \( \sigma^2_0 \). Note that \( \sigma^2_\eta \) and \( \gamma_1 \) can empirically be estimated from the sample equivalent of \( E \left[ X_t^2 \right] \) and \( E \left[ X_t X_{t-1} \right] \).

c) Third case : \( k = 2 \)

If the first and the second order of serial correlation are both statistically significant, we have 4 parameters to estimate \( \theta_0, \theta_1, \theta_2 \) and \( \sigma^2_\eta \) from the following system of equations :

\(^5\)The developments are presented in appendix C.
\[
\begin{align*}
E[X_t^2] &= (\theta_0^2 + \theta_1^2 + \theta_2^2)\sigma_\eta^2 \\
E[X_t X_{t-1}] &= (\theta_0 \theta_1 + \theta_1 \theta_2)\sigma_\eta^2 \\
E[X_t X_{t-1}] &= \theta_0 \theta_2 \sigma_\eta^2 \\
1 &= \theta_0 + \theta_1 + \theta_2 
\end{align*}
\] (37)

The resolution of this system of equations gives the following results:

\[
\sigma_\eta^2 = E[X_t^2] + 2E[X_t X_{t-1}] + 2E[X_t X_{t-2}] 
\] (38)

\[
\theta_1 = \frac{1}{2} \frac{\sqrt{1 - 4\delta_1}}{2} 
\] (39)

\[
\theta_0 = \frac{(1 - \theta_1)}{2} + \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2} 
\] (40)

\[
\theta_2 = \frac{(1 - \theta_1)}{2} - \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2} 
\] (41)

with

\[
\delta_1 = \frac{E[X_t X_{t-1}]}{\sigma_\eta^2} 
\] (42)

\[
\delta_2 = \frac{E[X_t X_{t-2}]}{\sigma_\eta^2} 
\] (43)

Then, the solutions of the system exist if and only if \( \delta_1 \leq \frac{1}{4} \) and \( \delta_2 \leq \frac{(1-\theta_1)^2}{4} \). To have satisfactory solutions, \( \delta_1 \) and \( \delta_2 \) should lead in these intervals:

\[
0 \leq \delta_1 \leq \frac{1}{4} 
\] (44)

\[
0 \leq \delta_2 \leq \frac{(1-\theta_1)^2}{4} 
\] (45)

The first and the second order of autocorrelation should neither be too high nor negative because if \( \delta_1 < 0 \) (i.e. if \( \text{Cov}(X_t, X_{t-1}) < 0 \) and/or if \( \delta_2 < 0 \) (\( \text{Cov}(X_t, X_{t-2}) < 0 \)), we will have \( \theta_1 < 0 \) and/or \( \theta_2 < 0 \) and there is a possibility that \( \theta_0 \) is superior to 1, and thereby \( \xi \) superior to 1. In other words, if the one of the serial correlation or if the both are negative, all the weights won’t be positive and the unsmoothing will be incongruous, because \( \xi \) will be higher than 1, and \( \sigma_\xi^2 \) will be lower than \( \sigma_\eta^2 \). Note that \( \sigma_\eta^2, \delta_1 \) and \( \delta_2 \) can empirically be estimated from the sample equivalent of \( E[X_t^2], E[X_t X_{t-1}] \) and \( E[X_t X_{t-2}] \).
d) Decision process

Before evaluating the pure persistence for each fund, we calculate the first and the second order of the serial correlation of returns and the decision process is as follows:

(*) If neither is statistically significant, we keep the observed returns.

(**) If only the first order serial correlation is significant (k=1), we estimate \(\sigma_0^2\) and \(\gamma_1\), and:
- If \(0 \leq \gamma_1 \leq \frac{1}{4}\), we estimate \(\theta_0\), \(\theta_1\) and the unsmoothed returns as follows:
  \[
  R_t^c = \frac{R_t^o - \hat{\theta}_1 R_{t-1}^c}{\theta_0}
  \]
  (46)

  Note that if \(k = 1\), the estimation of the unsmoothed returns is based on the assumption that the first return is an unsmoothed return.

  - \(\gamma_1 < 0\) implies that it is not possible to have satisfactory solutions and we exclude the fund from our sample.
  - \(\gamma_1 > \frac{1}{4}\) implies that the first order of serial correlation is high, and therefore we estimate the model as if \(k = 2\) to see whether we can have a solution. If not, we exclude the fund from our sample.

(***) If both the first and second order serial correlation are statistically significant, we estimate \(\sigma_0^2\), \(\delta_1\), \(\delta_2\) and \(\theta_1\), and check whether \(0 \leq \delta_1 \leq \frac{1}{4}\) and \(0 \leq \delta_2 \leq \frac{(1-\theta_1)^2}{4}\). In that case, we estimate \(\theta_0\), \(\theta_2\) and the unsmoothed returns as follows:
  \[
  R_t^c = \frac{R_t^o - \hat{\theta}_1 R_{t-1}^c - \hat{\theta}_2 R_{t-2}^c}{\theta_0}
  \]
  (47)

  If \(\delta_1\) and \(\delta_2\) are not comprised in these intervals, we exclude the fund from our sample because either we can not have satisfactory solutions or we can not simply have solutions.

4. Data

Our hedge funds data comprises monthly net of fee returns of 7255 live and dead funds provided by Hedge Fund Research Inc. (HFR) and covers the period starting from January 1994 through December 2007. However, we excluded funds with less than 36 consecutive monthly returns in order to estimate the pure persistence with sufficient data. This led us to a total of 4783 funds. Our data consists of 20 hedge fund strategies and is representative of hedge fund universe. Table 1 exhibits the statistics of funds for different strategies and the values presented are the average values across the strategies. We can see that there is an unequal distribution of funds in various strategies. Funds of funds are most numerous (1748), whereas Short selling has the lowest number of funds (13). On average, all the strategies exhibit a positive mean with the highest values for Emerging market (1.81%), Sector (1.44%) and Equity non hedge (1.37%).
Table 1: Descriptive statistics of hedge fund returns

<table>
<thead>
<tr>
<th>Category</th>
<th>Mean (%)</th>
<th>Vol. (%)</th>
<th>Skew</th>
<th>Kurt</th>
<th>Number of funds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.67</td>
<td>1.57</td>
<td>-0.46</td>
<td>5.24</td>
<td>92</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>1.11</td>
<td>2.31</td>
<td>0.33</td>
<td>6.41</td>
<td>107</td>
</tr>
<tr>
<td>Emerging Mkts</td>
<td>1.81</td>
<td>5.06</td>
<td>0.06</td>
<td>5.96</td>
<td>196</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>1.09</td>
<td>3.55</td>
<td>0.19</td>
<td>5.18</td>
<td>992</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.63</td>
<td>2.13</td>
<td>-0.18</td>
<td>6.06</td>
<td>193</td>
</tr>
<tr>
<td>Equity Non-H.</td>
<td>1.37</td>
<td>5.19</td>
<td>0.06</td>
<td>5.13</td>
<td>121</td>
</tr>
<tr>
<td>Event Driven</td>
<td>1.13</td>
<td>2.94</td>
<td>0.05</td>
<td>6.17</td>
<td>174</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.54</td>
<td>1.68</td>
<td>-0.44</td>
<td>9.77</td>
<td>70</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.66</td>
<td>3.35</td>
<td>0.26</td>
<td>4.62</td>
<td>21</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.60</td>
<td>1.88</td>
<td>-0.55</td>
<td>8.93</td>
<td>65</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.69</td>
<td>1.89</td>
<td>-1.42</td>
<td>13.30</td>
<td>50</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.79</td>
<td>1.94</td>
<td>-1.66</td>
<td>20.46</td>
<td>38</td>
</tr>
<tr>
<td>FOF</td>
<td>0.70</td>
<td>1.68</td>
<td>-0.37</td>
<td>5.13</td>
<td>1747</td>
</tr>
<tr>
<td>Macro</td>
<td>0.96</td>
<td>3.62</td>
<td>0.07</td>
<td>5.06</td>
<td>212</td>
</tr>
<tr>
<td>Market Timing</td>
<td>1.05</td>
<td>3.88</td>
<td>0.64</td>
<td>7.70</td>
<td>24</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.97</td>
<td>5.03</td>
<td>0.29</td>
<td>4.55</td>
<td>223</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.77</td>
<td>1.54</td>
<td>0.05</td>
<td>7.63</td>
<td>43</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.91</td>
<td>1.98</td>
<td>-0.04</td>
<td>6.69</td>
<td>211</td>
</tr>
<tr>
<td>Sector</td>
<td>1.44</td>
<td>4.92</td>
<td>0.30</td>
<td>6.07</td>
<td>191</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.12</td>
<td>6.83</td>
<td>0.00</td>
<td>6.23</td>
<td>13</td>
</tr>
<tr>
<td>All</td>
<td>0.93</td>
<td>2.80</td>
<td>-0.11</td>
<td>5.72</td>
<td>4783</td>
</tr>
</tbody>
</table>

Short selling, Equity non-hedge and Emerging market exhibit the highest values in volatility. As regards the third and the fourth moment of the distribution, hedge funds exhibit skewed returns and excess kurtosis. These descriptive statistics are in line with the results found in various studies which have documented the non-normality of hedge fund returns (Fung and Hsieh (1997), Liang (2000), etc.).

It is also well documented that hedge fund data is subject to different biases, such as survivorship bias or backfill bias. We construct our data set so as to limit any exposure to these biases. By using the returns of live and dead funds, we avoid the survivorship bias given that the persistence is evaluated for successful and unsuccessful funds. In order to account for the backfill bias, some studies exclude the first 12 monthly returns, because some funds might report their returns before their inclusion in the database if those returns are good. To verify whether it was necessary to use the same process with our sample, we estimated for each fund the difference in mean with and without the first 12 months. The values obtained are presented in table 2. \( \mu_{(\text{all})} - \mu_{(\text{minus 12})} \) is the difference between the mean of the whole set of the fund returns and that excluding the first 12 months. The average differences for each strategy and the corresponding t-statistic are presented in the table.
Table 2: Average differences in mean with and without the first 12 monthly returns

<table>
<thead>
<tr>
<th>Strategy</th>
<th>μ(all) - μ(minus 12) (%)</th>
<th>t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.064</td>
<td>1.15</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.044</td>
<td>0.48</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>-0.003</td>
<td>-0.02</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.069</td>
<td>2.33</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.057</td>
<td>1.44</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.091</td>
<td>0.86</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.051</td>
<td>0.71</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.056</td>
<td>0.65</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.037</td>
<td>0.25</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.052</td>
<td>0.93</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.051</td>
<td>0.54</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.090</td>
<td>0.99</td>
</tr>
<tr>
<td>FOF</td>
<td>-0.015</td>
<td>-1.33</td>
</tr>
<tr>
<td>Macro</td>
<td>0.02</td>
<td>0.35</td>
</tr>
<tr>
<td>Market Timing</td>
<td>-0.003</td>
<td>-0.02</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.022</td>
<td>0.30</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.039</td>
<td>0.34</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.076</td>
<td>1.28</td>
</tr>
<tr>
<td>Sector</td>
<td>0.069</td>
<td>0.75</td>
</tr>
<tr>
<td>Short Selling</td>
<td>-0.046</td>
<td>-0.34</td>
</tr>
</tbody>
</table>

We can see that the differences in mean are small and for some strategies they are negative (Emerging market, FOF, Market timing Short selling), which means that it does not necessary increase the mean when one includes the first 12 months of data. The spreads vary from a minimum -0.046% for Short selling to a maximum of 0.091% for Equity non-hedge. The t-statistics show that the spreads are not statistically different from zero, except for Equity hedge funds. Including the first 12 months of returns does not necessarily create a backfill bias in our database, and we will use all available in our study.

5. Estimation results

5.1 Serial correlation of hedge fund returns

Before proceeding to the unsmoothing of returns, we analyse first the serial correlation of hedge funds in our data. Table 3 presents the first and the second order serial correlation of the reported returns across strategies. Columns 5 and 9 present for each strategy, the percentage of funds exhibiting a statistical significant serial correlation of order 1 or 2.
Table 3: Serial correlation of order 1 and 2 for reported returns

<table>
<thead>
<tr>
<th>Sector</th>
<th>$\rho_1$ Mean</th>
<th>$\rho_1$ Min.</th>
<th>$\rho_1$ Max.</th>
<th>Sign. at 5% level (%)</th>
<th>$\rho_2$ Mean</th>
<th>$\rho_2$ Min.</th>
<th>$\rho_2$ Max.</th>
<th>Sign. at 5% level (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convert. Arb</td>
<td>0.38</td>
<td>-0.07</td>
<td>0.86</td>
<td>90.2</td>
<td>0.12</td>
<td>-0.22</td>
<td>0.81</td>
<td>23.9</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.25</td>
<td>-0.17</td>
<td>0.55</td>
<td>65.4</td>
<td>0.08</td>
<td>-0.24</td>
<td>0.52</td>
<td>15.0</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.12</td>
<td>-0.22</td>
<td>0.49</td>
<td>24.5</td>
<td>0.01</td>
<td>-0.26</td>
<td>0.32</td>
<td>3.1</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.11</td>
<td>-0.35</td>
<td>0.71</td>
<td>19.7</td>
<td>0.01</td>
<td>-0.52</td>
<td>0.47</td>
<td>6.7</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.06</td>
<td>-0.32</td>
<td>0.88</td>
<td>17.1</td>
<td>0.00</td>
<td>-0.42</td>
<td>0.85</td>
<td>6.7</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.10</td>
<td>-0.34</td>
<td>0.37</td>
<td>18.2</td>
<td>0.00</td>
<td>-0.24</td>
<td>0.35</td>
<td>3.3</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.20</td>
<td>-0.33</td>
<td>0.51</td>
<td>47.7</td>
<td>0.05</td>
<td>-0.23</td>
<td>0.35</td>
<td>8.6</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.15</td>
<td>-0.43</td>
<td>0.76</td>
<td>34.3</td>
<td>0.03</td>
<td>-0.53</td>
<td>0.61</td>
<td>11.4</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.20</td>
<td>-0.07</td>
<td>0.33</td>
<td>57.1</td>
<td>0.07</td>
<td>-0.11</td>
<td>0.30</td>
<td>19.0</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.13</td>
<td>-0.46</td>
<td>0.83</td>
<td>24.6</td>
<td>-0.05</td>
<td>-0.35</td>
<td>0.80</td>
<td>7.7</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.28</td>
<td>-0.05</td>
<td>0.50</td>
<td>56.0</td>
<td>0.01</td>
<td>-0.27</td>
<td>0.22</td>
<td>0.0</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.22</td>
<td>-0.07</td>
<td>0.57</td>
<td>44.7</td>
<td>0.15</td>
<td>-0.12</td>
<td>0.50</td>
<td>28.9</td>
</tr>
<tr>
<td>FOF</td>
<td>0.17</td>
<td>-0.57</td>
<td>0.66</td>
<td>29.4</td>
<td>-0.03</td>
<td>-0.35</td>
<td>0.48</td>
<td>4.2</td>
</tr>
<tr>
<td>Macro</td>
<td>0.06</td>
<td>-0.29</td>
<td>0.40</td>
<td>11.3</td>
<td>-0.03</td>
<td>-0.36</td>
<td>0.30</td>
<td>2.8</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.08</td>
<td>-0.15</td>
<td>0.39</td>
<td>25.0</td>
<td>0.06</td>
<td>-0.20</td>
<td>0.37</td>
<td>25.0</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.03</td>
<td>-0.30</td>
<td>0.52</td>
<td>6.7</td>
<td>-0.09</td>
<td>-0.44</td>
<td>0.42</td>
<td>2.2</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.17</td>
<td>-0.26</td>
<td>0.49</td>
<td>37.2</td>
<td>0.08</td>
<td>-0.12</td>
<td>0.38</td>
<td>16.3</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.18</td>
<td>-0.40</td>
<td>0.84</td>
<td>42.7</td>
<td>0.03</td>
<td>-0.35</td>
<td>0.65</td>
<td>11.4</td>
</tr>
<tr>
<td>Sector</td>
<td>0.09</td>
<td>-0.24</td>
<td>0.62</td>
<td>15.2</td>
<td>-0.02</td>
<td>-0.36</td>
<td>0.48</td>
<td>8.4</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.06</td>
<td>-0.13</td>
<td>0.34</td>
<td>15.4</td>
<td>-0.06</td>
<td>-0.15</td>
<td>0.18</td>
<td>0.0</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>-0.006</td>
<td>-</td>
<td>-</td>
<td>-0.04</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

On average, Convertible arbitrage, Distress securities, Fixed income convertible bonds, Fixed income high yield, Fixed income mortgage exhibit a higher first order serial correlation. These strategies exhibit also the higher proportions of funds with a statistical significant serial correlation. And even if the second order serial correlation is on average lower across all strategies, it is higher for these previous strategies which are generally invested in illiquid securities. Therefore, one can expect that the unsmoothing process may apply for most of the funds in these strategies.

Note also that the serial correlation profile can vary a lot among funds in each strategy and the gap between the lowest and the highest serial correlation can be very high. In some strategies, there exist some funds whose first or second order serial correlation is higher than 0.80 (Convertible arbitrage, Equity market neutral, Fixed income diversified, Relative value arbitrage). This shows that if one wants accurate results in analyzing hedge funds, it is important to work on a fund by fund basis rather than analysing the aggregate data of indices. Table 3 also shows that strategies involved in more liquid securities like Macro or Managed futures are those for which the first order serial correlation is lower. Therefore, the unsmoothing process should apply less for these strategies.

The last row of the table shows the first and the second order serial correlation of S&P500 monthly returns from January 1994 to December 2007. We can see that they are very small and not statistically
significant.

5.2. Results of the unsmoothing of returns

For the purpose of comparison, we proceed to the unsmoothing in two ways. First, we impose first and second order serial correlation to all funds (constrained model as done by GLM) and second, we unsmooth the returns according to the level of serial correlation of each fund (unconstrained model). Table 4 presents the average values of $\theta_0$, $\theta_1$, $\theta_2$ and $\xi$ for each strategy in the constrained model. The last column presents the percentage of funds for which we can have possible solutions (but not necessarily satisfactory solutions). Funds for which we couldn’t have possible solutions are those whose the level of first or second order of serial correlation is very high or the order of serial correlation is higher than 2.

Table 4: Constrained model

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$\theta_0$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\xi$</th>
<th>% of funds selected</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.64</td>
<td>0.27</td>
<td>0.09</td>
<td>0.52</td>
<td>96.7</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.76</td>
<td>0.18</td>
<td>0.06</td>
<td>0.65</td>
<td>96.3</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.91</td>
<td>0.09</td>
<td>-0.01</td>
<td>0.91</td>
<td>100.0</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.94</td>
<td>0.08</td>
<td>-0.02</td>
<td>1.01</td>
<td>99.1</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>1.18</td>
<td>-0.04</td>
<td>-0.14</td>
<td>10.65</td>
<td>99.5</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.94</td>
<td>0.07</td>
<td>-0.01</td>
<td>0.97</td>
<td>100.0</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.81</td>
<td>0.15</td>
<td>0.04</td>
<td>0.73</td>
<td>100.0</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.91</td>
<td>0.07</td>
<td>0.01</td>
<td>1.03</td>
<td>92.9</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.79</td>
<td>0.16</td>
<td>0.05</td>
<td>0.69</td>
<td>100.0</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>1.02</td>
<td>0.05</td>
<td>-0.07</td>
<td>1.36</td>
<td>90.8</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.77</td>
<td>0.24</td>
<td>-0.01</td>
<td>0.70</td>
<td>100.0</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.75</td>
<td>0.14</td>
<td>0.11</td>
<td>0.65</td>
<td>92.1</td>
</tr>
<tr>
<td>FOF</td>
<td>0.90</td>
<td>0.15</td>
<td>-0.05</td>
<td>0.91</td>
<td>98.9</td>
</tr>
<tr>
<td>Macro</td>
<td>1.04</td>
<td>0.02</td>
<td>-0.06</td>
<td>1.33</td>
<td>98.1</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.93</td>
<td>0.04</td>
<td>0.03</td>
<td>0.99</td>
<td>100.0</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>1.13</td>
<td>0.01</td>
<td>-0.15</td>
<td>1.47</td>
<td>99.1</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.83</td>
<td>0.11</td>
<td>0.06</td>
<td>0.77</td>
<td>100.0</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.89</td>
<td>0.10</td>
<td>0.01</td>
<td>0.96</td>
<td>91.9</td>
</tr>
<tr>
<td>Sector</td>
<td>1.01</td>
<td>0.06</td>
<td>-0.07</td>
<td>1.26</td>
<td>99.0</td>
</tr>
<tr>
<td>Short Selling</td>
<td>1.00</td>
<td>0.07</td>
<td>-0.07</td>
<td>1.05</td>
<td>100.0</td>
</tr>
<tr>
<td>All</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>98.2</td>
</tr>
</tbody>
</table>

By constraining the GLM model to be an MA(2), we can see that this could lead to unsatisfactory results. Indeed, for some strategies, we have on average negative values (weights) for $\theta_1$ and $\theta_2$, and the consequences are more undesirable for the most liquid strategies. Especially for Equity market neutral, Macro, Managed futures, Short selling and for Fixed income diversified, which have on average a value of $\theta_0$ higher or equal to one. GLM (2004) obtained similar results for some strategies of their
database\textsuperscript{6}. This leads to a smoothing index $\xi > 1$, and therefore to a lower volatility of unsmoothed returns, which is contrary to the hypothesis of the model. Then, for those strategies, the unsmoothing process will lead to an underestimation of the risk adjusted performance of the funds. However, for more illiquid strategies, imposing an MA(2) model does not necessarily raise this problem. The average value of $\theta_0$ for Convertible arbitrage, Distress securities, Fixed income convertible bonds, Fixed income high yield, and Fixed income mortgage is lower than one and their smoothing index is also lower than one.

In table 5, we present the results for the second approach where we don't constrain the model to be an MA(2). Column seven shows for each strategy the percentage of funds exhibiting no statistically significant serial correlation. Column eight shows the percentage for which only the first order serial correlation is statistically significant and column nine shows the percentage for which both the first and the second order serial correlation are statistically significant.

<table>
<thead>
<tr>
<th>Table 5: Unconstrained model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_0$</td>
</tr>
<tr>
<td>Convert. Arb</td>
</tr>
<tr>
<td>Distress Sec.</td>
</tr>
<tr>
<td>Emerging Mkt</td>
</tr>
<tr>
<td>Equity Hedge</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
</tr>
<tr>
<td>Equity Non H.</td>
</tr>
<tr>
<td>Event Driven</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
</tr>
<tr>
<td>FOF</td>
</tr>
<tr>
<td>Macro</td>
</tr>
<tr>
<td>Market Timing</td>
</tr>
<tr>
<td>Managed Fut.</td>
</tr>
<tr>
<td>Merger Arb.</td>
</tr>
<tr>
<td>Relative Value</td>
</tr>
<tr>
<td>Sector</td>
</tr>
<tr>
<td>Short Selling</td>
</tr>
<tr>
<td>Total HF</td>
</tr>
</tbody>
</table>

The unsmoothing process is then applied according to the case of each fund. We remind that one of the objectives of this study is to compare the pure persistence of hedge funds across strategies for

\textsuperscript{6}They used returns of 909 hedge funds from TASS database. The period of estimation starts from November 1977 to January 2001. HFR and TASS database don't have the same classification for hedge funds, but in their study, Equity hedge, Macro, Managed futures and Short selling are among strategies which exhibited a value of $\theta_0$ higher to one and/or negative values for $\theta_1$ or $\theta_2$. 

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smoothed and unsmoothed returns. Therefore, we should have the same number of funds when comparing the smoothed and unsmoothed returns for a strategy, and when it is not possible to unsmooth the returns of a fund, this one is excluded. Fortunately, we can see that we do not exclude a lot of funds; from 4783 funds of the sample, we only exclude 1.8%. The percentage of exclusion is of course not the same across strategies; it is more than 10% only for Fixed income diversified (10.8%), but this strategy doesn’t have an important weight in the sample. There is only 7 funds excluded from this strategy.

In column seven, we see that it is not necessary to unsmooth returns for the majority of funds for liquid strategies. Indeed, for Equity hedge, Equity market neutral, Equity non hedge, Macro, Managed futures, Sector and Short selling almost 80% of funds or more don’t need to be unsmoothed because their serial correlation is not statistically significant. This is not the case for illiquid strategies where Convertible arbitrage, Fixed income convertible bonds, Fixed income mortgage exhibit a significant percentage of funds which have to be unsmoothed up to lag 2. We also see that with the unconstrained model, we always have satisfactory solution because it takes into account the level of serial correlation of the fund. It is also interesting to notice that for Macro, Managed futures and Short selling funds there is no need to unsmooth returns up to lag 2.

5.3. On persistence of hedge fund returns

Table 6 compares for each strategy, the average positive persistence for funds with no serial correlation and for those of which it is necessary to unsmooth returns. Column 3 and 5 show the proportion of funds which exhibit a statistically significant positive persistence at the 5% level. We can note that funds with smoothed returns have on average a higher level of positive persistence (except for Emerging market), and the difference could be high. We also note that there are more funds exhibiting a statistically significant positive persistence in the universe of smoothed returns funds than in the universe of non smoothed returns funds. These results suggest that the smoothing of returns can contribute to increase the positive persistence. It is nevertheless important to notice that the majority of funds of almost all strategies (except for Managed futures and Short selling) exhibit statistically significant positive persistence as well as for smoothed than for unsmoothed returns.

To verify whether the smoothing can contribute to increase the positive persistence of returns, we evaluated the persistence of smoothed and unsmoothed returns of funds exhibiting a statistically significant serial correlation of returns.

\[ t = \frac{\bar{r}_{1:n} - 0.5}{1/\sqrt{n}} > 1.645 \]

where \( n \) is the number of monthly returns for that fund.
Table 6: Positive persistence for funds with no serial correlation and for funds with one or second order serial correlation

<table>
<thead>
<tr>
<th>Funds with k = 0</th>
<th>Funds with k = 1 or k = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p11</td>
</tr>
<tr>
<td>Convertible Arb</td>
<td>0.73</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.79</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.73</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.67</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.67</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.66</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.73</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.77</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.64</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.73</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.84</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.84</td>
</tr>
<tr>
<td>FOF</td>
<td>0.75</td>
</tr>
<tr>
<td>Macro</td>
<td>0.65</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.62</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.59</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.77</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.76</td>
</tr>
<tr>
<td>Sector</td>
<td>0.68</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.55</td>
</tr>
</tbody>
</table>

(*) Proportion of funds with p11 significantly > 0.5 at 5% level

The results are presented in table 7 where we observe that for these funds the average positive persistence drops considerably when one unsmooths the returns. The average drop of positive persistence accross strategies varies between -9.1% for Market timing and -25.4% for Short selling, even if the persistence is not statistically significant for any fund of the latter. We also observe, accross all strategies, a decrease in the percentage of funds exhibiting a statistically significant positive persistence at the 5% level. Distress securities, Fixed income high yield, Fixed income mortgage and Funds of funds exhibit the highest proportions of funds with a statistically significant positive persistence. Managed futures, Macro and Short selling have the lower proportion of funds with statistically significant positive persistence. Another important point to mention here is that for almost all strategies, the average positive persistence of unsmoothed returns for funds with k= 1 or 2, becomes lower than the average positive persistence for funds which no serial correlation (Table 6). The exception comes from Fixed income diversified (0.75 vs 0.73) and Market timing (0.79 vs 0.62).
Table 7: Positive persistence of smoothed and unsmoothed returns for funds with k=1 or k=2

<table>
<thead>
<tr>
<th>Smoothed returns</th>
<th>Unsmoothed returns</th>
<th>Variation of p11 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p11</td>
<td>Signif. &gt; 0.5 (%)</td>
</tr>
<tr>
<td>Convertible Arb</td>
<td>0.84</td>
<td>98.8</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.84</td>
<td>100.0</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.73</td>
<td>97.9</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.72</td>
<td>91.5</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.72</td>
<td>87.5</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.72</td>
<td>95.5</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.80</td>
<td>96.4</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.80</td>
<td>85.7</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.66</td>
<td>50.0</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.85</td>
<td>100.0</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.85</td>
<td>100.0</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.91</td>
<td>100.0</td>
</tr>
<tr>
<td>FOF</td>
<td>0.80</td>
<td>99.4</td>
</tr>
<tr>
<td>Macro</td>
<td>0.68</td>
<td>86.4</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.87</td>
<td>100.0</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.60</td>
<td>36.4</td>
</tr>
<tr>
<td>Merger Arbs.</td>
<td>0.82</td>
<td>100.0</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.84</td>
<td>98.8</td>
</tr>
<tr>
<td>Sector</td>
<td>0.72</td>
<td>89.3</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.62</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Overall, our findings suggest the smoothing of returns (voluntary or involuntary) is done at the advantage of the manager given that it contributes to increase the persistence of his positive returns.

If we aggregate the positive persistence of returns for funds with no serial correlation and the positive persistence of unsmoothed returns for funds with serial correlation, we obtain the following results (table 8) which may represent the average “true” positive persistence for each strategy. With aggregate data, the majority of funds for most strategies exhibit statistically significant positive persistence at the 5% level (17 out of 20 strategies). At the 1% level, it is the case for 9 strategies of which arbitrage strategies, fixed income strategies, FOF and other strategies based on illiquid securities (Convertible arbitrage, Distress securities, Event driven, Fixed income arbitrage, Fixed income high yield, Fixed income mortgage, FOF, Merger arbitrage and Relative value arbitrage). The lowest values of persistence are for Short selling (0.54), Managed futures (0.58) and Fixed income convertible bonds (0.59). The highest are for Fixed income mortgage (0.82), Fixed income high yield (0.76) and some arbitrage strategies.
Table 8: Positive persistence of “true” returns for all funds

<table>
<thead>
<tr>
<th></th>
<th>$p_{11}$</th>
<th>Signif. &gt; 0.5 at 5% (%)</th>
<th>Signif. &gt; 0.5 at 1% (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.67</td>
<td>69.7</td>
<td>51.7</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.74</td>
<td>83.8</td>
<td>72.4</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.70</td>
<td>77.9</td>
<td>44.1</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.66</td>
<td>65.0</td>
<td>35.7</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.66</td>
<td>60.0</td>
<td>34.1</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.65</td>
<td>63.0</td>
<td>41.2</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.71</td>
<td>82.1</td>
<td>63.0</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.74</td>
<td>74.2</td>
<td>66.7</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.59</td>
<td>38.1</td>
<td>23.8</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.73</td>
<td>79.3</td>
<td>50.0</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.76</td>
<td>90.0</td>
<td>72.0</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.82</td>
<td>94.4</td>
<td>91.7</td>
</tr>
<tr>
<td>FOF</td>
<td>0.74</td>
<td>88.9</td>
<td>71.0</td>
</tr>
<tr>
<td>Macro</td>
<td>0.64</td>
<td>52.2</td>
<td>28.8</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.67</td>
<td>75.0</td>
<td>45.8</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.58</td>
<td>25.9</td>
<td>11.6</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.74</td>
<td>88.4</td>
<td>79.1</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.74</td>
<td>76.8</td>
<td>64.6</td>
</tr>
<tr>
<td>Sector</td>
<td>0.67</td>
<td>60.6</td>
<td>35.1</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.54</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

5.4. Persistence vs probability of positive returns

Positive persistence evaluates a manager’s capability to deliver consecutive positive returns. The approach focuses on each past positive return and observes the sign of the following one. Although this information is relevant, it does not necessarily provide insight about the odds to deliver positive or negative returns. For that purpose, we should estimate the unconditional probability of positive returns, $P_1$, which takes into account the number of positive returns during the evaluation period. To support our assertion, let us look at the following example. Suppose a manager whose performance over 10 periods is as follows, where 1 represents the occurrence of a positive return and 0 that of a non-positive:

0 0 0 0 0 1 1 1 0 0

The probability of positive returns and the positive persistence can be estimated by respectively counting the number of 1s and the number of subsequent 1. In this case, $P_1 = 3/10 = 0.3$, and $p_{11} = 2/3 = 0.66$. This can be interpreted as a positive persistence, but a low performance on a regular basis (low value of $P_1$)\(^8\). Then, looking only at $p_{11}$ is misleading in evaluating the overall performance of the

\(^8\)Here, we don’t take into account the level of returns.
manager. Another look at this example shows that there is also the presence of negative persistence. In fact, if there is positive persistence and negative persistence, a high value of \( p_{11} \) will not be an indication of a high probability to have positive returns. On the other hand, if there is positive persistence and no negative persistence, the values of \( p_{11} \) and \( P_1 \) should not be very different and a high positive persistence will be an indication of a high probability of positive returns. Table 9 presents the average values of \( p_{11}, p_{00} \) and \( P_1 \) for hedge funds strategies.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>( p_{11} )</th>
<th>( p_{00} )</th>
<th>( P_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.67</td>
<td>0.32</td>
<td>0.68</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.74</td>
<td>0.30</td>
<td>0.73</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.70</td>
<td>0.33</td>
<td>0.69</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.66</td>
<td>0.37</td>
<td>0.65</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.66</td>
<td>0.36</td>
<td>0.65</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.65</td>
<td>0.41</td>
<td>0.63</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.71</td>
<td>0.33</td>
<td>0.70</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.74</td>
<td>0.28</td>
<td>0.74</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.59</td>
<td>0.41</td>
<td>0.59</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.73</td>
<td>0.33</td>
<td>0.72</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.76</td>
<td>0.30</td>
<td>0.75</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.82</td>
<td>0.26</td>
<td>0.81</td>
</tr>
<tr>
<td>FOF</td>
<td>0.74</td>
<td>0.36</td>
<td>0.71</td>
</tr>
<tr>
<td>Macro</td>
<td>0.64</td>
<td>0.37</td>
<td>0.64</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.67</td>
<td>0.37</td>
<td>0.65</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.58</td>
<td>0.43</td>
<td>0.58</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.74</td>
<td>0.25</td>
<td>0.75</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.74</td>
<td>0.29</td>
<td>0.73</td>
</tr>
<tr>
<td>Sector</td>
<td>0.67</td>
<td>0.37</td>
<td>0.66</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.54</td>
<td>0.56</td>
<td>0.49</td>
</tr>
</tbody>
</table>

We can see that for almost all strategies there is no negative persistence except for Short selling funds of which about 15% of funds (2 out of 13) have statistically significant value of \( p_{00} > 0.5 \) at the 5% level. This means that a monthly loss is generally followed by a gain in the hedge funds universe. Column 6 shows the probability of positive returns. We see that in general, the values of \( p_{11} \) are not very different from those of \( P_1 \); this is because of the absence of negative persistence of returns in the hedge funds universe. The last column shows the percentage of funds for which the probability
of positive returns is statistically superior to 0.5 at the 5% level\(^9\). We can note that except for Short selling, Managed futures and Fixed income convertible bonds, the majority of funds have a probability of positive returns superior to 0.5. The highest proportion is for Merger arbitrage where all the funds present a statistically significant probability of delivering positive returns. It is followed by Fixed income mortgage (97.22%) and Fixed income high yield (96%).

On the basis of these results, we can conclude that in spite of a context where markets faced difficult periods since years 2000, hedge funds have been able to deliver positive returns and this, in a sustainable way. Arbitrage strategies and some fixed income strategies seem to be more prone to deliver absolute returns.

5.5. On persistence with respect to a high water mark

Although the results obtained above are interesting, the measures used unfortunately don’t account for the level of returns. It is important to note that the absence of negative persistence in hedge funds returns \((p_{00} \text{ not statistically superior to 0.5})\), doesn’t mean that in the case of a loss, the capital will be recovered the next period (month), but simply that after a loss there is a strong probability that the returns is positive during the next month. It is important to seize the fact that the fund’s capability to recover losses in the subsequent period depends on both the size of the loss and the manager’s ability to generate positive returns of the same amplitude. Therefore, when a fund faces a large drawdown, it will require an important profit in the subsequent period or several small successive profits to recover the capital lost. This aspect is relevant for hedge funds because it is well known that several strategies, in particular arbitrage strategies generate positive returns of small amplitude in general, but when they face losses, they are often larger in amplitude. It can often take several periods before the funds recover the capital lost. Taking into account the level of returns also gives an indication on the evolution of the high water mark of a manager. Most hedge funds are subject a high water mark criterion, which means that the manager will only receive performance fees on that particular pool of invested money when its value exceeds its previous maximum value. By accounting for the level of returns, we can estimate the “performance with respect to the high water mark” which can be defined as the probability of increasing the high water mark and the “persistence with respect to the high water mark” which can be defined as the probability of increasing the high water mark the next period given that it has been increased the current period. These estimations are performance measures in the sense that they give the frequency at which a manager is able to receive performance fees\(^10\).

\(^9\)To get those values we calculated the confidence interval of \(\hat{P}_{1,i}\) for each fund \(i\), with

\[
\hat{P}_{1,i} = \frac{n_{+,i}}{n_{i}}
\]

where \(n_{+,i}\) is the number of positive returns and \(n_{i}\) is the size of sample for fund \(i\). By the central limit theorem, we have:

\[
\Pr \left( \frac{(n_{+,i}/n_{i}) - 1.96}{\sqrt{(n_{+,i}/n_{i})(1 - (n_{+,i}/n_{i}))}} \right) < P_{1,i} < \frac{(n_{+,i}/n_{i}) + 1.96}{\sqrt{(n_{+,i}/n_{i})(1 - (n_{+,i}/n_{i}))}} \right) = 0.95
\]

If the lower bound of this confidence interval is superior to 0.5, the probability is statistically superior to 0.5 at 5% level. We can notice that the smaller \(n_{i}\) is, the larger the confidence interval is.

\(^{10}\)Here, we assume a hurdle rate of 0% given that it is the value generally applied by hedge fund managers.
Let us define $C_t$ and $H_t$ respectively as the cumulative wealth and the high water mark at time $t$. Then:

$$
C_t = C_{t-1}(1 + r_t)
$$
$$
H_t = \max(C_t, H_{t-1})
$$

with $C_0$ and $H_0$ normalised at 1$.

Let also define the dichotomus variable $I'_t$ which takes the following values :

$$
I'_t = 1 \quad \text{if} \quad H_t > H_{t-1}
$$
$$
I'_t = 0 \quad \text{if} \quad H_t = H_{t-1}
$$

By this process, and as in the preceding model, we can obtain the following probabilities:

$$
P'_1 = \Pr[I'_t = 1]
$$
$$
p'_{11} = \Pr[I'_{t+1} = 1|I'_t = 1]
$$
$$
p'_0 = \Pr[I'_{t+1} = 0|I'_t = 0]
$$

$P'_1$ is the probability of increasing the high water mark and it could also be defined as the probability of receiving performance fees. $p'_{11}$ is the persistence in increasing the high water mark or the probability of increasing the high water mark the next period given that it has been increased the current period; it could also be defined as the persistence in the reception of performance fees. $p'_0$ is the persistence in the stagnation of the high water mark or the probability of having the same high water mark the next period given that it doesn’t change the current period; it could also be defined as the persistence in the absence of performance fees.

It is important to note that these measures are settled for an investment made at the inception date of the fund and enable to compare all the funds on the basis of their performance since their beginning. Indeed, a manager has a different high water mark for each investment made at a different time. Thus, when $I'_t = 1$, it means that the manager receives performance fees from an investor who invested money at time $t = 0$ and when $I'_t = 0$, it means that he doesn’t receive any performance from an investor who invested at that time. However, $I'_t = 1$ also means that the manager receives performance fees from all investors who have invested from time $0$ to time $t-1$, and $I'_t = 0$ also means that he doesn’t receive any performance fees from all investors who have invested during this period, but it could receive performance fees from some investors who have invested between time $1$ and time $t-1$.

Table 10 shows the values of the three probabilities for each strategy. We see first of all that the values of $p'_{11}$ are not very different from those of $p_{11}$ (table 9) (this could be interpreted as a similarity in the persistence of positive returns and the persistence in the reception of performance fees). But this doesn’t mean that measuring the persistence with returns or with respect to a high water mark is equivalent.
Table 10: Positive and negative persistence with respect to the high water mark and probability of increasing the high water mark (“true” returns)

<table>
<thead>
<tr>
<th></th>
<th>$p'_{11}$</th>
<th>Signif. &gt; 0.5 (%)</th>
<th>$p'_{00}$</th>
<th>Signif. &gt; 0.5 (%)</th>
<th>$P'_{1}$</th>
<th>Signif. &gt; 0.5 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.68</td>
<td>74.2</td>
<td>0.69</td>
<td>75.3</td>
<td>0.48</td>
<td>18.0</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.74</td>
<td>82.9</td>
<td>0.65</td>
<td>59.0</td>
<td>0.57</td>
<td>35.2</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.69</td>
<td>70.8</td>
<td>0.72</td>
<td>67.2</td>
<td>0.46</td>
<td>16.4</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.66</td>
<td>63.8</td>
<td>0.75</td>
<td>79.4</td>
<td>0.41</td>
<td>9.2</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.67</td>
<td>62.7</td>
<td>0.73</td>
<td>73.0</td>
<td>0.43</td>
<td>11.4</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.64</td>
<td>60.5</td>
<td>0.79</td>
<td>87.4</td>
<td>0.36</td>
<td>1.7</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.71</td>
<td>82.1</td>
<td>0.69</td>
<td>72.8</td>
<td>0.50</td>
<td>23.1</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.74</td>
<td>78.8</td>
<td>0.62</td>
<td>45.5</td>
<td>0.60</td>
<td>42.4</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.56</td>
<td>42.9</td>
<td>0.83</td>
<td>95.2</td>
<td>0.28</td>
<td>4.8</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.75</td>
<td>81.0</td>
<td>0.69</td>
<td>74.1</td>
<td>0.54</td>
<td>29.3</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.76</td>
<td>90.0</td>
<td>0.64</td>
<td>58.0</td>
<td>0.61</td>
<td>48.0</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.83</td>
<td>94.4</td>
<td>0.62</td>
<td>47.2</td>
<td>0.66</td>
<td>75.0</td>
</tr>
<tr>
<td>FOF</td>
<td>0.73</td>
<td>84.4</td>
<td>0.71</td>
<td>73.8</td>
<td>0.51</td>
<td>23.8</td>
</tr>
<tr>
<td>Macro</td>
<td>0.64</td>
<td>53.2</td>
<td>0.78</td>
<td>88.8</td>
<td>0.38</td>
<td>3.9</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.62</td>
<td>58.3</td>
<td>0.75</td>
<td>87.5</td>
<td>0.40</td>
<td>16.7</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.54</td>
<td>29.6</td>
<td>0.84</td>
<td>94.4</td>
<td>0.26</td>
<td>2.3</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.76</td>
<td>93.0</td>
<td>0.66</td>
<td>69.8</td>
<td>0.58</td>
<td>44.2</td>
</tr>
<tr>
<td>Relative Value</td>
<td>0.75</td>
<td>80.3</td>
<td>0.65</td>
<td>60.6</td>
<td>0.57</td>
<td>43.9</td>
</tr>
<tr>
<td>Sector</td>
<td>0.66</td>
<td>58.5</td>
<td>0.74</td>
<td>78.7</td>
<td>0.42</td>
<td>11.7</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.53</td>
<td>23.1</td>
<td>0.97</td>
<td>100.0</td>
<td>0.06</td>
<td>0.0</td>
</tr>
</tbody>
</table>

In fact, when we look at the values of $p'_{00}$ and $P'_{1}$, we see that they are different from those of $p_{00}$ and $P_{1}$. Indeed, whereas the probability that a loss is followed by another loss is low for all strategies, the probability that a non-payment of performance fees from all investors is followed by another non-payment of performance fees is generally high. The fact that the values of $p'_{11}$ and $p'_{00}$ are statistically superior to 0.5 for the majority of funds in almost all strategies means that when a manager receives performance fees in a period, there is a high probability that he receives performance fees the next period; but it also means that when he doesn’t receive any performance fees in a period, there also a high probability that he receives no performance fees the next period because he will not be able to recover the capital lost at that period. These results are in line with what we mentioned before i.e. hedge fund managers carry out positive returns of small amplitude in general, but when they face losses, they are often of larger amplitudes and the managers are not able to recover the capital lost quickly; In terms of persistence, this translates in small increases of high water mark in good periods and a stagnation of the high water mark after a bad period.

Another important point is to notice that strategies where there is a higher persistence in increasing the high water mark are those where there is a lower persistence in the stagnation of the high water mark.
mark notably Fixed income mortgage (0.83 vs 0.62), Fixed income high yield (0.76 vs 0.64) and Merger arbitrage (0.76 vs 0.66). And vice versa, notably Short selling (0.53 vs 0.97), Managed futures (0.54 vs 0.84) and Fixed income convertible bonds (0.56 vs 0.83). These results seem to show that for strategies like Merger arbitrage and others which exhibit a higher positive persistence with respect to a high water mark, managers show more ability to bring the capital back to a value superior or equal to that preceding the loss. Is it because they have superior skills? It is difficult to answer to this question. But we can also address to question to know whether these strategies exhibit a shorter time to recover the capital lost. To this end, we should make a thorough examination given that the previous estimations concern the high water marks from all investments. Therefore, some investments made at different moments could be recovered at a certain time but not others, and the “overall” high water mark will not increase, and this will translate in a stagnation of the “overall” high water mark.

For instance, for an investment made at the inception date, the manager could have a certain high water mark at time t and receive a new investment at the end of time t + 1. At that moment, he could face a loss followed by another loss at time t +2 and a gain at time t + 3. The capital of the investment made at time t+1 could be recovered at time t + 3 and the manager could receive performance fees from this investment, but this doesn’t necessarily mean that he will also recover the capital lost at time t+1 (from the investment made at the inception date). Then, a better way to have an idea on the ability to recover the capital after a loss is to estimate the recovery time for each loss and to see the average for each fund and for each strategy. This issue will be addressed in the next section.

Back to column 6 of table 10 which shows the unconditional probability of increasing the high water mark, we see that contrary to the results of table 9, $p'_{11}$ and $P'_1$ are not similar as $p_{11}$ and $P_1$ were; this is because of high values of $p'_{00}$ which indicate an inverse persistence. The relatively low values of $P'_1$ for the majority of strategies (they are inferior to 0.60 for 17 out of 20 strategies) mean that the managers are not able to increase their high water mark on a regular basis even if they have in general a high probability to deliver positive returns (see values of $P_1$). Nevertheless, Fixed income mortgage and Fixed income high yield managers are more prone to increase their high water mark, whereas Short selling, Managed futures and fixed income convertible bonds managers are less prone to increase their high water mark.

From these results, we can state that even if hedge funds are able to deliver positive (absolute) returns, they have more difficulties to increase their high water mark on a regular basis. Indeed, periods of small and consecutive increases of the high water mark are often interrupted by periods of stagnation of the high water mark, and this is due to their risk exposure which can lead to important drawdowns in bad periods. It is important to note that when we use the words “important drawdowns” it is does not mean that hedge funds hold high risky positions leading to large losses, but simply that the losses can be important in comparison to gains. The specific risk-return profile of many hedge fund strategies characterized by payoffs similar to that of short puts on market indices has been mentioned by several studies (Fung and Hsieh (1997), Mitchell and Pulvino (2001), Agarwal and Naik (2004)). This option like-payoff can be modeled via a covered call\textsuperscript{11}. A covered call is a strategy in which an investor writes a call option contract while at the same time owning an equivalent number of shares

\textsuperscript{11}We take the example of a covered call in order to have a strategy that combines the trading of assets and derivatives given that hedge funds are general invested in traditional assets (stocks, bonds, etc.) as well as derivatives.
of the underlying stock. While this strategy can offer limited protection from a decline in price of the underlying stock and limited profit participation with an increase in stock price, it generates income because the investor keeps the premium received from writing the call. Thus, the investor will have a profit lower than that of the underlying stock if the latter increases substantially (the option will be exercised) and will have lower loss than the underlying stock. We are not implying that many hedge funds use covered call strategies, but using this kind of strategy can give a payoff similar to that of hedge funds. Indeed, looking at the historical performance of some hedge fund strategies we observe that they have payoffs similar to that of a covered call on S&P500 index i.e. positive returns are not large in general and losses are of lower amplitude than the S&P500. A covered call can help a manager aiming at providing absolute returns because even if it limits gains, it can contribute to increase the regularity of these gains. Absolute returns do not necessarily mean high returns, but “good returns” regardless on the direction of the market. On the other hand, even if the strategy helps to reduce losses the latter could be substantial in comparison with gains, because the effect on gains and losses is not symmetrical and this gives a payoff with important drawdowns in comparison to gains.

5.6. Average time to recover the capital after a loss

The previous results showed that when a hedge fund manager faces a loss, it could take a certain time before he recovers the capital preceding this loss. These measures of performance and persistence with respect to a high water mark give an indication on the performance of a manager, but also an indication on the risk that an investor could face when he invests in hedge funds. Indeed, if an investor plans to withdraw his money after a loss, he should know that there is a low probability that in the following period, the manager brings it back a level superior or equal to that preceding the loss. Therefore, the investor should wait for a certain time if he wants to withdraw a capital superior or equal to the last high water mark of the manager. Then, in order to evaluate the good time to withdraw his money, he should take into account this aspect and the advance notice of the fund which varies from one fund to another. Given the previous results, we estimated the average time to recover the capital after a loss. For this purpose, we calculated for each recovered loss, the number of months necessary to come back to a capital superior or equal to that preceding the loss. Table 11 shows the results for all strategies. The values are averaged for each fund and thereafter averaged for each strategy. Columns 2 to 5 show respectively the mean, the 25th percentile, the 75th percentile and the volatility of the average recovery time by strategy. Column 6 shows the average proportion of losses for each strategy. This statistic helps to see the frequency of losses by strategy on the same basis given that funds of different strategies don’t have the same lifespan.
Table 11: Average time to recover the capital after a loss (in months)

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>25th prct.</th>
<th>75th prct.</th>
<th>Volatility</th>
<th>Average proportion of losses (%)</th>
<th>Number of funds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conv arb</td>
<td>3.99</td>
<td>2.87</td>
<td>4.75</td>
<td>1.61</td>
<td>27.83</td>
<td>19.94</td>
</tr>
<tr>
<td>Dist sec</td>
<td>3.00</td>
<td>2.17</td>
<td>3.37</td>
<td>1.44</td>
<td>24.67</td>
<td>28.05</td>
</tr>
<tr>
<td>Emes Mkt</td>
<td>4.73</td>
<td>2.36</td>
<td>5.21</td>
<td>3.65</td>
<td>25.78</td>
<td>28.39</td>
</tr>
<tr>
<td>Equi Hedge</td>
<td>4.17</td>
<td>2.54</td>
<td>4.76</td>
<td>2.57</td>
<td>28.82</td>
<td>30.39</td>
</tr>
<tr>
<td>Equi mkt neut</td>
<td>3.66</td>
<td>2.32</td>
<td>4.22</td>
<td>2.03</td>
<td>26.17</td>
<td>30.39</td>
</tr>
<tr>
<td>Equi non hedge</td>
<td>5.00</td>
<td>2.81</td>
<td>5.77</td>
<td>3.17</td>
<td>35.48</td>
<td>30.39</td>
</tr>
<tr>
<td>Event Driven</td>
<td>3.40</td>
<td>2.18</td>
<td>3.92</td>
<td>2.12</td>
<td>27.89</td>
<td>20.29</td>
</tr>
<tr>
<td>Fixed inc arb.</td>
<td>2.60</td>
<td>1.73</td>
<td>3.31</td>
<td>1.27</td>
<td>17.21</td>
<td>19.30</td>
</tr>
<tr>
<td>Fixed inc Con</td>
<td>7.15</td>
<td>3.46</td>
<td>10.41</td>
<td>4.31</td>
<td>44.14</td>
<td>20.29</td>
</tr>
<tr>
<td>Fixed inc Div</td>
<td>3.03</td>
<td>1.97</td>
<td>3.97</td>
<td>1.45</td>
<td>20.29</td>
<td>19.30</td>
</tr>
<tr>
<td>Fixed inc Hig</td>
<td>2.49</td>
<td>1.75</td>
<td>2.88</td>
<td>1.03</td>
<td>18.94</td>
<td>19.30</td>
</tr>
<tr>
<td>Fixed inc Mor</td>
<td>4.14</td>
<td>1.75</td>
<td>6.73</td>
<td>3.19</td>
<td>18.94</td>
<td>19.30</td>
</tr>
<tr>
<td>FOF</td>
<td>3.43</td>
<td>2.43</td>
<td>3.74</td>
<td>1.85</td>
<td>22.50</td>
<td>19.30</td>
</tr>
<tr>
<td>Macro</td>
<td>4.25</td>
<td>2.91</td>
<td>4.87</td>
<td>2.23</td>
<td>30.54</td>
<td>19.30</td>
</tr>
<tr>
<td>Mkt timing</td>
<td>5.21</td>
<td>2.81</td>
<td>7.09</td>
<td>3.37</td>
<td>37.21</td>
<td>19.30</td>
</tr>
<tr>
<td>CTA</td>
<td>4.63</td>
<td>3.46</td>
<td>5.46</td>
<td>1.85</td>
<td>39.69</td>
<td>20.46</td>
</tr>
<tr>
<td>Merger arb.</td>
<td>3.18</td>
<td>2.33</td>
<td>4.04</td>
<td>1.24</td>
<td>28.56</td>
<td>20.46</td>
</tr>
<tr>
<td>Rel value</td>
<td>2.94</td>
<td>1.98</td>
<td>3.38</td>
<td>1.63</td>
<td>19.94</td>
<td>19.30</td>
</tr>
<tr>
<td>Sector</td>
<td>4.14</td>
<td>2.41</td>
<td>5.00</td>
<td>2.32</td>
<td>28.05</td>
<td>19.30</td>
</tr>
<tr>
<td>Short selling</td>
<td>7.12</td>
<td>5.73</td>
<td>8.10</td>
<td>2.33</td>
<td>60.54</td>
<td>19.30</td>
</tr>
</tbody>
</table>

We note that for hedge fund strategies, the average time to recover a capital lost is more than 3 months even if for some strategies, managers recover it earlier (Fixed income high yield (2.49), Fixed income arbitrage (2.60), Relative value arbitrage (2.94)). The explanation could be that either managers of those strategies don’t face great losses in general, or they take a lot of risk after a loss the recover the capital quickly. For some other strategies it takes more time to recover the capital after a loss, notably for Fixed income convertible bonds (7.15), Short selling (7.12) and Market timing (5.21). One should expect that funds exhibiting higher positive persistence should take less time to recover the capital, but it is not necessarily the case. The negative relation between the time to recover the capital after a loss and the positive persistence seems to be more obvious for funds exhibiting lower positive persistence of returns and with respect to a high water mark. Short selling, Fixed income convertible bonds and Managed futures funds are among those taking in general more time to recover the capital after a loss. However, for funds exhibiting higher positive persistence the relation is verified for only Fixed income high yield. Merger arbitrage funds come at the sixth position in terms of time to recover the capital and Fixed income mortgage funds which exhibit the highest positive persistence of returns and with respect of a high water mark come at the tenth position. One reason could be that Fixed income mortgage funds exhibit negative outliers more than other categories of funds (they exhibit the lowest (-1.66) asymmetry and the highest kurtosis (20.46)). Concerning the proportion of
losses, hedge funds exhibit in general fewer losses than gains, except for short selling where the number of losses and gains is almost the same. The lowest proportions are generally for fixed income strategies and for Managed futures.

These results show how the advance notice imposed by most hedge funds constitutes not only a good protection against withdrawals from investors needing liquidity but also from unhappy investors after a loss. Indeed, given the asymmetry in the amplitude of gains and losses, it is important for a manager to set up some delay for withdrawals of money, especially investors attempting to withdraw after a loss. An advance notice superior to the average time to recover the capital may enable the manager to bring the capital back to its value before the loss allowing the investor to change his mind. The fact that the average time to recover a loss is more than 3 months for most strategies suggests that a median advance notice of 30 days is not necessarily optimal for hedge fund managers. Managers having an advance notice superior to 3 months will probably have more chance to keep unhappy investors willing to withdraw their money after a loss. However, due to competition between managers, it may be difficult to fix large periods of advance notice, even if in our data the maximum advance notice is one year.

Table 11 shows the average recovery time for losses which have been recovered. It is important to mention that in our sample some losses have not yet been recovered, so they are discarded in table 11. The unrecovered losses are not exclusively large losses, but also losses that occurred near the end of our sample period. Table 12 exhibits the statistics for non recovered losses for each strategy. Column 2 presents the average proportion of non recovered losses and Column 3 shows the average proportion of large losses among these non recovered losses. Large losses are those which in absolute value are higher than 2 standard deviations of the distribution of returns. We can see that Short selling and almost all fixed income strategies (Fixed income high yield, Fixed income diversified and Fixed income mortgage) among those which exhibit the highest proportions of non recovered losses. However, contrary to these fixed income strategies, the non-recovering of Short selling losses is not due in a considerable proportion to severe drawdowns. Indeed, the average proportion of large losses in non recovered losses is only 1.92%, whereas it varies between 15% and 24% for the concerned fixed income strategies. We can also notice that losses are mostly recovered for Emerging market, Market timing and Merger arbitrage strategies. Table 11 gives a good insight of the average time to recover a loss, but these results must be interpreted with some caution given that for some strategies like fixed income strategies many losses have not been recovered. Another interesting point is that the recovery period for losses can be quite significant. For example, the maximum recovery time is greater than 100 months for some managers (Equity hedge (115), Macro (114), Emerging market (113), FOF (111), Equity non hedge (110). We note that many of these drawdowns occured between August 97 and July 98. In some cases, the individual losses were not very large, however the managers were unable to generate sufficient subsequent positive returns prior to enduring a further drawdown. The second half of 1998 were not a good period for hedge fund industry or for themarket in general and it was thereby a difficult period to recover prior losses. For instance, an investor who would have invested 1$ in the particular Equity hedge fund that exhibited the longest time to recover, would have waited for 115 months before the breaking even. This also means that the manager would have not received
any performance fees from this investor for 115 months. However, this doesn’t mean that the manager 
would have not received performance fees from other investors who entered the fund at a later date.

These findings do not augur well for investors in the forthcoming months. Indeed, given that the 
current financial crisis may have more negative impacts on hedge fund industry than the 1998 crisis, 
one should expect that some losses incurred during 2008, might need more than 100 months to be 
recovered. However, it is important to note that the recovery times found above 100 months can be 
considered as outliers.

Table 12: Statistics for non recovered losses

<table>
<thead>
<tr>
<th></th>
<th>Proportion of non recovered losses (%)</th>
<th>Proportion of large losses in non recovered losses (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conv arb</td>
<td>9.63</td>
<td>13.33</td>
</tr>
<tr>
<td>Dist sec</td>
<td>11.88</td>
<td>7.33</td>
</tr>
<tr>
<td>Eme Mkt</td>
<td>5.60</td>
<td>5.78</td>
</tr>
<tr>
<td>Equi Hedge</td>
<td>9.79</td>
<td>10.35</td>
</tr>
<tr>
<td>Equi mkt neut</td>
<td>10.96</td>
<td>12.36</td>
</tr>
<tr>
<td>Equi non hedge</td>
<td>9.92</td>
<td>5.37</td>
</tr>
<tr>
<td>Event Driven</td>
<td>10.76</td>
<td>13.42</td>
</tr>
<tr>
<td>Fixed inc arb.</td>
<td>11.43</td>
<td>21.00</td>
</tr>
<tr>
<td>Fixed inc Con</td>
<td>9.95</td>
<td>4.37</td>
</tr>
<tr>
<td>Fixed inc Div</td>
<td>13.50</td>
<td>18.62</td>
</tr>
<tr>
<td>Fixed inc Hig</td>
<td>16.76</td>
<td>23.67</td>
</tr>
<tr>
<td>Fixed inc Mor</td>
<td>11.93</td>
<td>15.37</td>
</tr>
<tr>
<td>FOF</td>
<td>8.51</td>
<td>11.83</td>
</tr>
<tr>
<td>Macro</td>
<td>8.77</td>
<td>7.63</td>
</tr>
<tr>
<td>Mkt timing</td>
<td>6.05</td>
<td>8.65</td>
</tr>
<tr>
<td>CTA</td>
<td>8.80</td>
<td>5.34</td>
</tr>
<tr>
<td>Merger arb.</td>
<td>6.15</td>
<td>21.51</td>
</tr>
<tr>
<td>Rel value</td>
<td>11.52</td>
<td>17.89</td>
</tr>
<tr>
<td>Sector</td>
<td>8.12</td>
<td>7.54</td>
</tr>
<tr>
<td>Short selling</td>
<td>23.95</td>
<td>1.92</td>
</tr>
</tbody>
</table>

5.7. On relative persistence

We also try to investigate the relative persistence in the performance of hedge funds of our sample 
to see whether we capture the same effect when we evaluate the two kinds of persistence (pure vs 
relative). Given that it is not the main objective of our study, the investigation of relative persistence
is done here in a simply way by using raw returns. Our return measurement intervals are half-yearly and yearly. To this end, we use a non-parametric method by constructing a contingency table of winners and losers where a fund is a winner (W) if the periodic return of that fund is greater than the median return of all the funds following the same strategy in that period, otherwise it is a loser (L). Persistence is observed for funds that are winners in two consecutive periods (WW), or losers in two consecutive periods (LL). Winners in the first period and losers in the second period are denoted by WL, and losers in the first period and winners in the second period are denoted by LW. We use the cross-product ratio (CPR) to make our tests:

\[
CPR = \frac{WW \times LL}{WL \times LW}
\]  

The statistical significance of the CPR is determined by using the standard error of the natural logarithm of the CPR given in Christensen (1990):

\[
\sigma_{\ln(CPR)} = \sqrt{\frac{1}{WW} + \frac{1}{WL} + \frac{1}{WW} + \frac{1}{LW} + \frac{1}{LL}}
\]  

In order to have sufficient funds per strategy, we use data from January 2000 to December 2007, given that before this period some strategies don’t have enough funds to give accurate results for relative persistence. Funds which provided returns for the first interval and ceased to provide returns for all the months of the second interval were penalized by allowing a return of 0% for each month where we had missing informations. Indeed, we choose 0% because we think it is a good compromise given that we didn’t have information whether the fund was dead or whether it was still alive but didn’t want to publish its returns because it was closed to new investments. Table 13 shows the percentage of cases exhibiting statistically significant persistence in performance from January 2000 to December 2007. On a half-yearly basis, we have 15 sub-periods of estimation and on a yearly basis we have 7 sub-periods. We observe that some strategies exhibit more relative persistence than others. On a half-yearly basis, the presence of relative persistence is more important for FOF, Relative value, Equity hedge, Fixed income mortgage, Fixed income diversified funds whereas there is a lack of persistence for short selling, Merger arbitrage, Fixed income arbitrage, Managed future and Market timing funds. The results are quite similar on a yearly basis except that the relative persistence for equity based strategies is higher (Emerging market, Equity hedge, Equity non hedge, Equity market neutral). It is interesting to notice that there is a lack of relative persistence for some strategies exhibiting a higher level of positive persistence in returns and with respect to a high water mark namely Merger arbitrage and Fixed income high yield. Fixed income mortgage shows some relative persistence on a half-yearly basis, but a lack of persistence on a yearly basis. We also note that there is no relative persistence for funds exhibiting low level of positive persistence in returns and with respect to a high water mark namely Short selling, Managed futures and Fixed income convertible bonds. These results suggest that there is not necessarily a link between the pure persistence and the relative persistence. However, FOF and Equity based strategies seem to be the strategies where the relative persistence is more relevant, in other words where the choice of the manager is more relevant.
### Table 13: Relative persistence of hedge fund returns

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Half-yearly</th>
<th>Yearly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Convertible Arb</td>
<td>0.20</td>
<td>0.43</td>
</tr>
<tr>
<td>Distress Sec.</td>
<td>0.27</td>
<td>0.00</td>
</tr>
<tr>
<td>Emerging Mkt</td>
<td>0.47</td>
<td>0.71</td>
</tr>
<tr>
<td>Equity Hedge</td>
<td>0.60</td>
<td>0.71</td>
</tr>
<tr>
<td>Equity Mkt N.</td>
<td>0.40</td>
<td>0.57</td>
</tr>
<tr>
<td>Equity Non H.</td>
<td>0.33</td>
<td>0.57</td>
</tr>
<tr>
<td>Event Driven</td>
<td>0.67</td>
<td>0.43</td>
</tr>
<tr>
<td>Fixed Inc Arb.</td>
<td>0.07</td>
<td>0.00</td>
</tr>
<tr>
<td>Fixed Inc Con.</td>
<td>0.20</td>
<td>0.14</td>
</tr>
<tr>
<td>Fixed Inc Div</td>
<td>0.53</td>
<td>0.43</td>
</tr>
<tr>
<td>Fixed Inc Hig</td>
<td>0.27</td>
<td>0.29</td>
</tr>
<tr>
<td>Fixed Inc Mor</td>
<td>0.53</td>
<td>0.29</td>
</tr>
<tr>
<td>FOF</td>
<td>0.93</td>
<td>0.86</td>
</tr>
<tr>
<td>Macro</td>
<td>0.33</td>
<td>0.29</td>
</tr>
<tr>
<td>Market Timing</td>
<td>0.13</td>
<td>0.14</td>
</tr>
<tr>
<td>Managed Fut.</td>
<td>0.13</td>
<td>0.14</td>
</tr>
<tr>
<td>Merger Arb.</td>
<td>0.00</td>
<td>0.14</td>
</tr>
<tr>
<td>Relative Valuc</td>
<td>0.67</td>
<td>0.43</td>
</tr>
<tr>
<td>Sector</td>
<td>0.33</td>
<td>0.43</td>
</tr>
<tr>
<td>Short Selling</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

These results contrast slightly with those of Brown, Goetzmann and Ibbotson (1999), Agarwal and Naik (2000) and Liang (2000) who find the evidence of relative persistence at short term horizon but not at longer horizons. We find that the relative persistence does not necessarily decrease as the measurement horizon increases; on the contrary, for some strategies it increases and they exhibit more evidence of long term persistence than on short term persistence. Our results are in line with those of Jagannathan, Malakhov and Novikov (2006), and Kosowski, Naik and Teo (2007) who find evidence of long term relative persistence in hedge fund returns. Some reasons of these conflicting conclusions could be the difference in the methods of estimation. This could also be due to the differences in the data. The first three studies used data ending at 1998, whereas the two last studies and ours used data ending at December 2002, April 2005 and December 2007.

### 6. Conclusion

In this study, we have addressed the issue of hedge fund performance persistence using a model of Markov chains. Persistence is evaluated via transition probabilities, which make no a priori about the
distribution of returns. Persistence is also evaluated after accounting for serial correlation in hedge fund returns which is often due to the holding of illiquid assets or manager’s motivation to enhance his performance. For that purpose, we use a new approach based on the method of moments and on the model of Getzmannsky and al. (2004) to unsmooth returns. To assess the significance of persistence estimates, we also developed a t-test which accounts for the size of the sample of fund returns.

Our study shows first that the unsmoothing of returns is not necessary for all funds, especially for those of liquid strategies namely Macro, Managed futures, Sector and Short selling funds. Therefore, imposing an MA(2) model for all funds as done by Getzmannsky and al. (2004) could lead to incongruous results. We also note that the smoothing could contribute to increase the pure persistence of returns. Getzmannsky and al. (2004) have pointed that the evidence of relative persistence found in some studies may be indirectly linked to serial correlation in returns. Our results show that for almost all strategies, the average positive persistence of returns of funds with no statistically significant serial correlation is lower than that of funds with smoothed returns; and the average persistence of the latter drops considerably (between -9.1% and -25.4%) when one unsmooths returns. Nevertheless, our findings suggest that except for Short selling, Managed futures and Fixed income convertible bonds, the majority of funds of other strategies exhibit persistence of positive returns and almost all the funds exhibit no persistence of negative returns. Our results show that until 2007, hedge funds have been able to deliver absolute returns in a sustainable way in spite of periods of turbulence faced by the markets. It remains to see how the events of 2008 will affect these conclusions.

However, hedge funds exhibit difficulties in increasing their high water marks on a regular basis. Periods of consecutive positive returns are sometimes interrupted by some large drawdowns which take several periods to recover because the positive returns are generally smaller in size. This translates into positive and negative persistence with respect to a high water mark. In other words this leads to small and consecutive increases of the high water mark but also in stagnations of the high water mark over some periods. The estimated average time to recover the capital after a loss varies from 2.49 months (Fixed income high yield) to 7.15 months (Fixed income convertible bonds). Given that the current financial crisis will probably intensify the negative asymmetry of the distribution of hedge fund returns, the average time to recover losses will increase and with a median advance notice of 30 days, most of funds will not have enough time (flexibility) to reverse the situation in order to retain investors willing to withdraw their money; and this will accentuate the liquidation of funds as it has been the case recently. Many analysts foresee that about one-third of hedge funds could be liquidated because of the massive withdrawals from investors.

Our results also show that there is not necessarily a link between absolute persistence and relative persistence, therefore finding the evidence of one does not translate in the evidence of the other one.

These results raise the question about how an investor should evaluate the performance of a manager, especially in terms of pure persistence; and in the context of hedge funds where positive and negative returns could be of very different amplitudes, this is more relevant given that as long as the high water mark of the manager does not increase, the investor is not wealthier even if the manager delivers some positive returns.
References


APPENDIX

APPENDIX A: ESTIMATION OF $\sqrt{n} (\hat{p}_{11} - p_{11})$ BY THE DELTA METHOD

We know that $\hat{p}_{11}$ can also be expressed in the following way:

$$\hat{p}_{11} = \frac{\hat{P}_{11}}{\hat{P}_{11} + \hat{P}_{10}}$$

where $\hat{P}_{11} = \text{Pr}(I_t = 1; I_{t+1} = 1)$ and $\hat{P}_{10} = \text{Pr}(I_t = 1; I_{t+1} = 0)$ are jointed probabilities. Thus, $\hat{p}_{11}$ is a function of $\hat{P}_{11}$ and $\hat{P}_{10}$ and we can write:

$$\hat{p}_{11} = f(\hat{P}_{11}, \hat{P}_{10})$$

It is known that, for a given function $g$, the first-order Taylor series expansion of $g(x_0)$ around $x$ is:

$$g(x_0) = g(x) + g'(x)(x_0 - x)$$

$$\Rightarrow g(x_0) - g(x) = g'(x)(x_0 - x)$$

Then, we can write:

$$\hat{p}_{11} - p_{11} = \frac{\partial f}{\partial \hat{P}_{11}}(\hat{P}_{11}, \hat{P}_{10}) \cdot (\hat{P}_{11} - P_{11}) + \frac{\partial f}{\partial \hat{P}_{10}}(\hat{P}_{11}, \hat{P}_{10}) \cdot (\hat{P}_{10} - P_{10})$$

$$= \frac{(P_{11} + P_{10}) - P_{11}}{(P_{11} + P_{10})^2} (\hat{P}_{11} - P_{11}) + \frac{(-P_{11})}{(P_{11} + P_{10})^2} (\hat{P}_{10} - P_{10})$$

$$= \frac{P_{10}}{(P_{11} + P_{10})^2} (\hat{P}_{11} - P_{11}) - \frac{P_{11}}{(P_{11} + P_{10})^2} (\hat{P}_{10} - P_{10})$$

where $P_{11}$ and $P_{10}$ are the asymptotic jointed probabilities.

$$\Rightarrow \sqrt{n} (\hat{p}_{11} - p_{11}) = \sqrt{n} \frac{P_{10}}{(P_{11} + P_{10})^2} (\hat{P}_{11} - P_{11}) - \sqrt{n} \frac{P_{11}}{(P_{11} + P_{10})^2} (\hat{P}_{10} - P_{10})$$

Asymptotically we have$^{12}$:

$$P_{11} = P_{10} = \frac{1}{4}$$

$$\Rightarrow \sqrt{n} (\hat{p}_{11} - p_{11}) = \sqrt{n} (\hat{P}_{11} - P_{11}) - \sqrt{n} (\hat{P}_{10} - P_{10})$$

$^{12}$For $n \to \infty$, we can assume independance and the probabilites become

$$P_{11} = P_{10} = 1/4$$
Then:

\[ \text{Var} \left[ \sqrt{n} (\hat{p}_{11} - p_{11}) \right] = \text{Var} \left[ \sqrt{n} (\hat{p}_{11}) \right] + \text{Var} \left[ \sqrt{n} (\hat{P}_{10}) \right] - 2 \text{Cov} \left[ \sqrt{n} (\hat{P}_{11}) , \sqrt{n} (\hat{P}_{10}) \right] \]

**Estimation of** \( \text{Var} \left[ \sqrt{n} (\hat{P}_{11}) \right] \) **when** \( n \to \infty \)

\[
\text{Var} \left( \hat{P}_{11} \right) = E \left( \hat{P}_{11}^2 \right) - E \left( \hat{P}_{11} \right)^2 \\
E \left( \hat{P}_{11}^2 \right) = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 1, I_{i+1} = 1) \cdot (I_j = 1, I_{j+1} = 1) \right] = \frac{n}{n^2} \cdot \frac{1}{4}
\]

\[
E \left( \hat{P}_{11}^2 \right) = E \left[ \frac{1}{n^2} \sum_{i=2}^{n} \sum_{j=1}^{n} (I_i = 1, I_{i+1} = 1) \cdot (I_j = 1, I_{j+1} = 1) \right] = \frac{(n-1)}{n^2} \cdot \frac{1}{8}
\]

\[
E \left( \hat{P}_{11}^2 \right) = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 1, I_{i+1} = 1) \cdot (I_j = 1, I_{j+1} = 1) \right] = \frac{2 \sum_{j=2}^{n} (n-j)}{n^2} \cdot \frac{1}{16}
\]

\[ \Rightarrow E \left( \hat{P}_{11}^2 \right) = \frac{n}{4n^2} + \frac{2(n-1)}{8n^2} + \frac{2 \sum_{j=2}^{n} (n-j)}{16n^2} \]

\[ = \frac{n}{4n^2} + \frac{2(n-1)}{8n^2} + \frac{2(n-1)n - 2n(n+1)}{16n^2} \]

\[ = \frac{n^2 + 5n - 2}{16n^2} \]

Then:

\[
\text{Var} \left( \hat{P}_{11} \right) = E \left( \hat{P}_{11}^2 \right) - E \left( \hat{P}_{11} \right)^2 \\
= \frac{n^2 + 5n - 2}{16n^2} - \frac{1}{16} \\
= \frac{5n - 2}{16n^2}
\]
\[ \Rightarrow n \text{Var} \left( \hat{P}_{11} \right) = \text{Var} \left( \sqrt{n} \hat{P}_{11} \right) \]
\[ = \frac{5n - 2}{16n^2} \]
\[ = \frac{5}{16} - \frac{2}{16n} \]
\[ \Rightarrow \text{when } n \to \infty, \text{Var} \left( \sqrt{n} \hat{P}_{11} \right) \to \frac{5}{16} \]

**Estimation of \( \text{Var} \left[ \sqrt{n} \left( \hat{P}_{10} \right) \right] \) when \( n \to \infty \)**

\[ \text{Var} \left( \hat{P}_{10} \right) = E \left( \hat{P}_{10}^2 \right) - E \left( \hat{P}_{10} \right)^2 \]

\[ E \left( \hat{P}_{10}^2 \right) = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 1, I_{i+1} = 0) \cdot (I_j = 1, I_{j+1} = 0) \right] \]

The different cases are:

\[ j = i \quad ; \quad = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 1, I_{i+1} = 0) \cdot (I_i = 1, I_{i+1} = 0) \right] = \frac{n}{n^2} \frac{1}{4} \]

\[ j = i - 1 \quad ; \quad +E \left[ \frac{1}{n^2} \sum_{i=2}^{n} (I_i = 1, I_{i+1} = 0) \cdot (I_{i-1} = 1, I_i = 0) \right] = 0 \]

\[ j = i + 1 \quad ; \quad +E \left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 1, I_{i+1} = 0) \cdot (I_{i+1} = 1, I_{i+2} = 0) \right] = 0 \]

\[ |j - 1| > 1; \quad +E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{|j-i|>1} (I_i = 1, I_{i+1} = 0) \cdot (I_j = 1, I_{j+1} = 0) \right] = \frac{2 \sum_{j=2}^{n} (n - j)}{n^2} \frac{1}{16} \]

\[ \Rightarrow E \left( \hat{P}_{10}^2 \right) = \frac{n}{4n^2} + \frac{2 \sum_{j=2}^{n} (n - j)}{16n^2} \]
\[ = \frac{n^2 + n + 2}{16n^2} \]

Then:

\[ \text{Var} \left( \hat{P}_{10} \right) = E \left( \hat{P}_{10}^2 \right) - E \left( \hat{P}_{10} \right)^2 \]
\[ = \frac{n^2 + n + 2}{16n^2} - \frac{1}{16} \]
\[ = \frac{n + 2}{16n^2} \]
\[ n \text{Var} \left( \hat{P}_{10} \right) = \text{Var} \left( \sqrt{n} \hat{P}_{10} \right) \]

\[ = n \frac{(n + 2)}{16n^2} \]

\[ = \frac{1}{16} + \frac{2}{16n} \]

\[ \Rightarrow \text{When } n \to \infty, \text{Var} \left( \sqrt{n} \hat{P}_{10} \right) \to \frac{1}{16} \]

**Estimation of \( \text{Cov} \left( \sqrt{n} \hat{P}_{11}, \sqrt{n} \hat{P}_{10} \right) \) when \( n \to \infty $$

\[
\text{Cov} \left( \hat{P}_{11}, \hat{P}_{10} \right) = E \left( \hat{P}_{11} \hat{P}_{10} \right) - E \left( \hat{P}_{11} \right) E \left( \hat{P}_{10} \right)
\]

\[ E \left( \hat{P}_{11} \hat{P}_{10} \right) = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( I_i = 1, I_{i+1} = 1 \right) . I \left( I_j = 1, I_{j+1} = 0 \right) \right] \]

\[ j = i \quad ; \quad E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \left( I_i = 1, I_{i+1} = 1 \right) . I \left( I_i = 1, I_{i+1} = 0 \right) \right] = 0 \]

\[ j = i - 1 \quad ; \quad E \left[ \frac{1}{n^2} \sum_{i=2}^{n} \left( I_i = 1, I_{i+1} = 1 \right) . I \left( I_{i-1} = 1, I_i = 0 \right) \right] = 0 \]

\[ j = i + 1 \quad ; \quad E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \left( I_i = 1, I_{i+1} = 1 \right) . I \left( I_{i+1} = 1, I_{i+2} = 0 \right) \right] = \frac{(n - 1)}{n^2} \frac{1}{8} \]

\[ |j - 1| > 1; \quad E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{|j-i| > 1} \left( I_i = 1, I_{i+1} = 1 \right) . I \left( I_j = 1, I_{j+1} = 0 \right) \right] = \frac{2}{n^2} \sum_{j=2}^{n} \frac{(n - j)}{16n^2} \]

\[ \Rightarrow E \left( \hat{P}_{11} \hat{P}_{10} \right) = \frac{(n - 1)}{8n^2} + \frac{2}{16n^2} \frac{(n - j)}{n^2} \]

\[ = \frac{n - 1}{16n} \]

we have:

\[
\text{Cov} \left( \hat{P}_{11}, \hat{P}_{10} \right) = E \left( \hat{P}_{11} \hat{P}_{10} \right) - E \left( \hat{P}_{11} \right) E \left( \hat{P}_{10} \right)
\]

\[ = \frac{n - 1}{16n} - \frac{1}{16} \]

\[ = -\frac{1}{16} \]
\[ n \text{Cov} \left( \hat{P}_{11}, \hat{P}_{10} \right) = \text{Cov} \left( \sqrt{n} \hat{P}_{11}, \sqrt{n} \hat{P}_{10} \right) \]
\[ = n \cdot \left( -\frac{1}{16n} \right) \]
\[ = -\frac{1}{16} \]

\[ \Rightarrow \text{When } n \to \infty, \text{Cov} \left( \sqrt{n} \hat{P}_{11}, \sqrt{n} \hat{P}_{10} \right) \to -\frac{1}{16} \]

Then:
\[ \text{Var} \left[ \sqrt{n} (\hat{p}_{11} - p_{11}) \right] = \text{Var} \left[ \sqrt{n} (\hat{P}_{11}) \right] + \text{Var} \left[ \sqrt{n} (\hat{P}_{10}) \right] - 2 \text{Cov} \left[ \sqrt{n} (\hat{P}_{11}), \sqrt{n} (\hat{P}_{10}) \right] \]
\[ = \frac{5}{16} + \frac{1}{16} - 2 \left( -\frac{1}{16} \right) \]
\[ = \frac{1}{2} \]

**APPENDIX B: ESTIMATION OF \( \sqrt{n} (\hat{p}_{00} - p_{00}) \) WHEN \( n \to \infty \)**

The same developments as before give:
\[ \text{Var} \left[ \sqrt{n} (\hat{p}_{00} - p_{00}) \right] = \text{Var} \left[ \sqrt{n} (\hat{P}_{00}) \right] + \text{Var} \left[ \sqrt{n} (\hat{P}_{01}) \right] - 2 \text{Cov} \left[ \sqrt{n} (\hat{P}_{00}), \sqrt{n} (\hat{P}_{01}) \right] \]

- **Estimation of \( \text{Var} \left[ \sqrt{n} (\hat{P}_{00}) \right] \)** when \( n \to \infty \)

\[ \text{Var} \left( \hat{P}_{00} \right) = E \left( \hat{P}_{00}^2 \right) - E \left( \hat{P}_{00} \right)^2 \]
\[ E \left( \hat{P}_{00}^2 \right) = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 0, I_{i+1} = 0) \cdot (I_j = 0, I_{j+1} = 0) \right] \]
\[ j = i; = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 0, I_{i+1} = 0) \cdot (I_i = 0, I_{i+1} = 0) \right] = \frac{n^2 - 4}{n^2} \]
\[ j = i - 1; +E \left[ \frac{1}{n^2} \sum_{i=2}^{n} (I_i = 0, I_{i+1} = 0) \cdot (I_{i-1} = 0, I_i = 0) \right] = \frac{(n - 1) 1}{8} \]
\[ j = i + 1; +E \left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 0, I_{i+1} = 0) \cdot (I_{i+1} = 0, I_{i+2} = 0) \right] = \frac{(n - 1) 1}{8} \]
\[ |j - 1| > 1; +E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{|j-i|>1} (I_i = 1, I_{i+1} = 0) \cdot (I_j = 1, I_{j+1} = 0) \right] = \frac{2}{n^2} \sum_{j=2}^{n} \frac{(n - j)}{16} \]

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\[ \Rightarrow E \left( \hat{P}_{00}^2 \right) = \frac{n^2}{4n^2} + \frac{2(n-1)}{8n^2} + \frac{2 \sum_{j=2}^{n} (n-j)}{16n^2} = \frac{n^2 + 5n - 2}{16n^2} \]

We have

\[ Var \left( \hat{P}_{00} \right) = E \left( \hat{P}_{00}^2 \right) - E \left( \hat{P}_{00} \right)^2 = \frac{n^2 + 5n - 2}{16n^2} - \frac{1}{16} = \frac{5n - 2}{16n^2} \]

\[ \Rightarrow n Var \left( \hat{P}_{00} \right) = Var \left( \sqrt{n} \hat{P}_{00} \right) = \frac{5}{16} \]

\[ \Rightarrow When \ n \rightarrow \infty, \ Var \left( \sqrt{n} \hat{P}_{00} \right) \rightarrow \frac{5}{16} \]

**Estimation of \( Var \left[ \sqrt{n} \left( \hat{P}_{01} \right) \right] \) when \( n \rightarrow \infty \)**

\[ Var \left( \hat{P}_{01} \right) = E \left( \hat{P}_{01}^2 \right) - E \left( \hat{P}_{01} \right)^2 \]

\[ E \left( \hat{P}_{01}^2 \right) = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 0, X_{i+1} = 1) \cdot (I_j = 0, I_{j+1} = 1) \right] \]

\[ j = i \; ; \; + E \left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 0, I_{i+1} = 1) \cdot (I_i = 0, I_{i+1} = 1) \right] = \frac{n}{n^2} \frac{1}{4} \]

\[ j = i - 1 \; ; \; + E \left[ \frac{1}{n^2} \sum_{i=2}^{n} (I_i = 0, X_{i+1} = 1) \cdot (I_{i-1} = 0, I_i = 1) \right] = 0 \]

\[ j = i + 1 \; ; \; + E \left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 0, I_{i+1} = 1) \cdot (I_{i+1} = 0, I_{i+2} = 1) \right] = 0 \]

\[ |j-1| > 1; \; + E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{|j-i| > 1} (I_i = 0, I_{i+1} = 1) \cdot (I_j = 0, I_{j+1} = 1) \right] = \frac{2 \sum_{j=2}^{n} (n-j)}{n^2} \frac{1}{16} \]
\[ \Rightarrow E\left(\widehat{P}_{01}^2\right) = \frac{n^2}{4n^2} + \frac{2}{16n^2} \sum_{j=2}^{n} (n - j) \]
\[ = \frac{n^2 + n + 2}{16n^2} \]

We have:

\[ Var\left(\widehat{P}_{01}\right) = E\left(\widehat{P}_{01}^2\right) - E\left(\widehat{P}_{01}\right)^2 \]
\[ = \frac{n^2 + n + 2}{16n^2} - \frac{1}{16} \]
\[ = \frac{n + 2}{16n^2} \]

\[ \Rightarrow n Var\left(\widehat{P}_{01}\right) = Var\left(\sqrt{n}\widehat{P}_{01}\right) \]
\[ = \frac{n(n + 2)}{16n^2} \]
\[ = \frac{1}{16} + \frac{2}{16n} \]

\[ \Rightarrow \text{When } n \to \infty, Var\left(\sqrt{n}\widehat{P}_{01}\right) \to \frac{1}{16} \]

**Estimation of** \( Cov\left(\sqrt{n}\widehat{P}_{00},\sqrt{n}\widehat{P}_{01}\right) \) when \( n \to \infty \)

\[
Cov\left(\hat{P}_{00},\hat{P}_{01}\right) = E\left(\hat{P}_{00}\hat{P}_{01}\right) - E\left(\hat{P}_{00}\right) E\left(\hat{P}_{01}\right)
\]

\[
E\left(\hat{P}_{00}\hat{P}_{01}\right) = E\left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 0, I_{i+1} = 0). (I_j = 0, I_{j+1} = 1) \right]
\]

\[
j = i \quad ; \quad = E\left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 0, I_{i+1} = 0). (I_i = 0, I_{i+1} = 1) \right] = 0
\]

\[
j = i - 1 \quad ; \quad = E\left[ \frac{1}{n^2} \sum_{i=2}^{n} (I_i = 0, I_{i+1} = 0). (I_{i-1} = 0, I_i = 1) \right] = 0
\]

\[
j = i + 1 \quad ; \quad = E\left[ \frac{1}{n^2} \sum_{i=1}^{n} (I_i = 0, I_{i+1} = 0). (I_{i+1} = 0, I_{i+2} = 1) \right] = \frac{(n - 1) \cdot 1}{n^2} = \frac{1}{8}
\]
\[ |j - 1| > 1; \quad E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (I_i = 0, I_{i+1} = 0) (I_j = 0, I_{j+1} = 1) \right] = \frac{2 \sum_{j=2}^{n} (n - j)}{n^2 \frac{1}{16}} \]

\[ \Rightarrow E(\hat{P}_{00}, \hat{P}_{01}) = \frac{(n - 1)}{8n^2} + \frac{2 \sum_{j=2}^{n} (n - j)}{16n^2} \]

\[ = \frac{n^2 - n}{16n^2} = \frac{n - 1}{16n} \]

We have

\[ Cov(\hat{P}_{00}, \hat{P}_{01}) = E(\hat{P}_{00}, \hat{P}_{01}) - E(\hat{P}_{00}) E(\hat{P}_{01}) \]

\[ = \frac{n - 1}{16n} - \frac{1}{16} \]

\[ = -\frac{1}{16n} \]

\[ \Rightarrow nCov(\hat{P}_{00}, \hat{P}_{01}) = Cov(\sqrt{n}\hat{P}_{00}, \sqrt{n}\hat{P}_{01}) \]

\[ = \frac{n.(-1/16n)}{16} \]

\[ = -\frac{1}{16} \]

\[ \Rightarrow When \quad n \rightarrow \infty, \quad Cov(\sqrt{n}\hat{P}_{00}, \sqrt{n}\hat{P}_{01}) \rightarrow -\frac{1}{16} \]

We saw that:

\[ Var \left[ \sqrt{n} (\hat{p}_{00} - p_{00}) \right] = Var \left[ \sqrt{n} (\hat{P}_{00}) \right] + Var \left[ \sqrt{n} (\hat{P}_{01}) \right] - 2Cov \left[ \sqrt{n} (\hat{P}_{00}), \sqrt{n} (\hat{P}_{01}) \right] \]

\[ = \frac{5}{16} + \frac{1}{16} - 2(-\frac{1}{16}) \]

\[ = \frac{1}{2} \]

**APPENDIX C: ESTIMATION OF \( \theta_s \)**

- The first order serial correlation is significative : \( k = 1 \)

If the first order of serial correlation is statistically significant and not the second one, we have 3 parameters to estimate \( \theta_0, \theta_1 \) and \( \sigma^2 \) from the following system of equations:

\[
\begin{cases}
E[X_t^2] = (\theta_0^2 + \theta_1^2)\sigma^2_\theta \\
E[X_t, X_{t-1}] = \theta_0 \theta_1 \sigma^2_\theta \\
1 = \theta_0 + \theta_1
\end{cases}
\]

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By replacing $\theta_1$ in the first two equations by its value $1 - \theta_0$, we get:

$$\begin{cases} 
E[X_t^2] = (\theta_0^2 + (1 - \theta_0)^2)\sigma^2_n \\
E[X_t \cdot X_{t-1}] = (\theta_0 - \theta_0^2)\sigma^2_n
\end{cases}$$

This leads us to:

$$\sigma^2_n = E[X_t^2] + 2E[X_t \cdot X_{t-1}]$$

We can thus empirically estimate $\sigma^2_n$ from the sample equivalent of $E[X_t^2]$ and $E[X_t \cdot X_{t-1}]$. The second equation implies that:

$$\frac{E[X_t \cdot X_{t-1}]}{E[X_t^2] + 2E[X_t \cdot X_{t-1}]} = \theta_0 - \theta_0^2$$

Let

$$\gamma_1 = \frac{E[X_t \cdot X_{t-1}]}{E[X_t^2] + 2E[X_t \cdot X_{t-1}]}$$

We get:

$$\theta_0^2 - \theta_0 + \gamma_1 = 0$$

This equation has two solutions:

$$\begin{cases} 
\theta_{0,1} = \frac{1}{2} + \frac{\sqrt{1 - 4\gamma_1}}{2} \\
\theta_{0,2} = \frac{1}{2} - \frac{\sqrt{1 - 4\gamma_1}}{2}
\end{cases}$$

This implies that a solution exists if and only if $\gamma_1 \leq \frac{1}{4}$.

Given that $\theta_0 \geq \theta_1$, and both sum to 1, $\theta_0$ is higher than $\frac{1}{2}$, then

$$\theta_0 = \frac{1}{2} + \frac{\sqrt{1 - 4\gamma_1}}{2}$$

and

$$\theta_1 = 1 - \theta_0 = \frac{1}{2} - \frac{\sqrt{1 - 4\gamma_1}}{2}$$

We see here that $\theta_0$ is positive, but $\theta_1$ could be negative in certain conditions. Indeed, $\theta_1 < 0$ if

$$\frac{1}{2} - \frac{\sqrt{1 - 4\gamma_1}}{2} < 0$$

$$\Rightarrow \gamma_1 < 0$$

Then $\gamma_1$ should be $\geq 0$ to ensure that we have positive weights. We saw that:

$$\gamma_1 = \frac{E[X_t \cdot X_{t-1}]}{\sigma^2_n}$$
The sign of $\gamma_1$ depends on the numerator. This means that if $E[X_t.X_{t-1}] < 0$, it implies $\theta_1 < 0$. We have

\[
E[X_t.X_{t-1}] = E[X_t].E[X_{t-1}] + Cov(X_t, X_{t-1})
= Cov(X_t, X_{t-1})
\]
given that $X_t$ are centered returns. Thereby if $Cov(X_t, X_{t-1}) < 0$ we will have $\theta_1 < 0$. In other words it means that if the serial correlation of order 1 is negative, all the weights won’t be positive and the unsmoothing will be incongruous, because $\xi$ will be higher than 1, and $\sigma_2^2$ will be lower than $\sigma_0^2$.

Overall to have satisfactory solutions, $\gamma_1$ should lead in this interval:

\[
0 < \gamma_1 \leq \frac{1}{4}
\]

The first order of autocorrelation should not be negative or should not be too high.

**The first and the second order of serial correlation are significative : $k = 2$**

If the first and the second order of serial correlation are both statistically significant, we have 4 parameters to estimate $\theta_0$, $\theta_1$, $\theta_2$ and $\sigma_0^2$ from the following system of equations:

\[
\begin{align*}
E[X_t^2] &= (\theta_0^2 + \theta_1^2 + \theta_2^2)\sigma_0^2 \\
E[X_t.X_{t-1}] &= (\theta_0\theta_1 + \theta_1\theta_2)\sigma_0^2 \\
E[X_t.X_{t-1}] &= \theta_0\theta_2\sigma_0^2 \\
1 &= \theta_0 + \theta_1 + \theta_2
\end{align*}
\]

The development of the equations gives:

\[
\sigma_0^2 = E[X_t^2] + 2E[X_t.X_{t-1}] + 2E[X_t.X_{t-2}]
\]

We can estimate $\sigma_0^2$ empirically from the sample equivalent of $E[X_t^2]$, $E[X_t.X_{t-1}]$ and $E[X_t.X_{t-2}]$. From the second equation we have:

\[
\frac{E[X_t.X_{t-1}]}{\sigma_0^2} = \theta_1 - \theta_1^2
\]

\[
\Rightarrow \frac{E[X_t.X_{t-1}]}{E[X_t^2] + 2E[X_t.X_{t-1}] + 2E[X_t.X_{t-2}]} = \theta_1 - \theta_1^2
\]

Let

\[
\delta_1 = \frac{E[X_t.X_{t-1}]}{E[X_t^2] + 2E[X_t.X_{t-1}] + 2E[X_t.X_{t-2}]}
\]
We get:

$$\theta_1^2 - \theta_1 + \delta_1 = 0$$

This equation has two solutions:

$$\begin{cases} 
\theta_{1,1} = \frac{1}{2} + \frac{\sqrt{1 - 4\delta_1}}{2} \\
\theta_{1,2} = \frac{1}{2} - \frac{\sqrt{1 - 4\delta_1}}{2}
\end{cases}$$

As pointed by GLM in the Proposition 3 of their model:

(i) \( \theta_1 < 1/2 \);
(ii) \( \theta_1 < 1 - 2\theta_2 \)

From (i), it follows that:

$$\theta_1 = \frac{1}{2} - \frac{\sqrt{1 - 4\delta_1}}{2}$$

We also see here that to have a satisfactory solution:

$$0 \leq \delta_1 \leq \frac{1}{4}$$

From the value of \( \theta_1 \) we can get \( \theta_0 \). From the third equation, we have:

$$\frac{E[X_{t}X_{t-2}]}{E[X_{t}^2] + 2E[X_{t}X_{t-1}] + 2E[X_{t}X_{t-2}]} = \theta_0 - \theta_0^2 - \theta_0\theta_1$$

Let

$$\delta_2 = \frac{E[X_{t}X_{t-2}]}{E[X_{t}^2] + 2E[X_{t}X_{t-1}] + 2E[X_{t}X_{t-2}]}$$

We get:

$$\theta_0^2 - (1 - \theta_1)\theta_0 + \delta_2 = 0$$

This equation has two solutions:

$$\begin{cases} 
\theta_{0,1} = \frac{(1 - \theta_1)}{2} + \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2} \\
\theta_{0,2} = \frac{(1 - \theta_1)}{2} - \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2}
\end{cases}$$

From (ii), we have:

$$\theta_1 < 1 - 2(1 - \theta_0 - \theta_1)$$

$$\Rightarrow \theta_0 > \frac{1 - \theta_1}{2}$$

Thus the solution for \( \theta_0 \) is:

$$\theta_0 = \frac{(1 - \theta_1)}{2} + \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2}$$
We also see here that we have a solution if and only if \( \delta_2 \leq \frac{(1-\theta_1)^2}{4} \).

Next, we obtain \( \theta_2 = 1 - \theta_0 - \theta_1 \). This gives us

\[
\theta_2 = \frac{(1 - \theta_1)}{2} - \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2}
\]

We can see that \( \theta_2 \) could be negative in certain conditions. Indeed \( \theta_2 < 0 \) if

\[
\frac{(1 - \theta_1)}{2} - \frac{\sqrt{(1 - \theta_1)^2 - 4\delta_2}}{2} < 0
\]

\[\Rightarrow \delta_2 < 0\]

Thus \( \delta_2 \) should be \( \geq 0 \) to ensure that we have positive value of \( \theta_2 \). We saw that:

\[
\delta_2 = \frac{E[X_t, X_{t-2}]}{\sigma_\theta^2}
\]

This means that if \( \text{Cov}(X_t, X_{t-2}) < 0 \), in other words if the second order of serial correlation is negative, we will have a negative value for \( \theta_2 \).

Overall to have satisfactory solutions, \( \delta_1 \) and \( \delta_2 \) should lead in these intervals:

\[
0 \leq \delta_1 \leq \frac{1}{4}
\]

\[
0 \leq \delta_2 \leq \frac{(1-\theta_1)^2}{4}
\]