Welfare comparisons when populations differ in sizes: an empirical analysis

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Novembre 2009

Preliminary draft

Abstract

Should a larger population exhibit more social welfare than a smaller population? In which cases a smaller population can dominate a larger one when focusing on welfare ranking? In a recent paper, we have developed some technical procedures to address these questions and to attempt to find some relevant answers. It considers welfare dominance based on critical-level generalized utilitarianism (CLGU) in addition to poverty ranking. CLGU is extended to include dominance tests based on different orders of dominance and involving possible choices of poverty lines and possible values for critical levels. A few simulation experiments show that the sizes of populations may be a great concern for welfare ranking. Some empirical applications of the procedures are made to estimate some lower and upper bounds of the critical levels, using Canadian Surveys of Consumer Finances (SCF) for 1976 and 1986, and Canadian Survey of Labour and Income Dynamics (SLID) for 2006. Asymptotic tests reveal a dominance of 2006 over 1976 and 1986.

Keywords: Critical-level utilitarianism dominance, poverty dominance, welfare dominance, critical-level estimation, Canada.

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1 Introduction

Should a larger population exhibit more social welfare than a smaller population? In which case a smaller population can dominate a larger one when focusing on welfare ranking? This paper addresses these questions and attempts to find some relevant answers.

Welfare and inequality comparisons are easily led when populations have the same size. However, this is not the case when sizes may differ. In that situation, the tradition has been to overcome the issue related to variations in populations sizes by calling on the replication invariance axiom. The replication invariance axiom claims that an income distribution and its \( k \)-fold replication, with \( k \) being any positive integer, give the same level of social welfare. Therefore, welfare or inequality should be assessed in per capita terms. Given that, most of the literature has usually avoided the problem of variable size by supposing that welfare comparisons in per capita terms should be sufficient. However, as Blackorby, Bossert, and Donaldson (2005) argued, populations sizes should sometimes matter when comparing welfare.

Our work develops this idea and follows the “critical-level generalized utilitarianism” (CLGU) principle of Blackorby and Donaldson (1984) which allows assessing if adding a person to an existing population can be considered as improving social welfare. Then using this principle, one can be able to order distributions that have not the same dimension. Hence, a particular interest of this paper is to compare the social welfare of two populations in the context of variable sizes. The motivation for welfare ranking involving different populations sizes lies on the fact that it is certainly the most generally encountered case in empirical analysis of welfare dominance. Recent works have tried to make welfare comparisons in the case where populations have unequal sizes without using the replication invariance axiom. Among them are Aboudi, Thon, and Wallace (2007) who use a generalization of the well-known concept of majorization to make inequality comparisons. Pogge (2007) has also adopted a different approach based on a Pareto criterion to compare welfare involving different number of individuals. But this approach does not consider the inequality between distributions.

This paper also aims at building empirical applications on the CLGU framework, using Canadian Survey of Consumer Finances (SCF) for 1976 and Canadian Survey of Labour and Income Dynamics (SLID) for 2006. To be more precise, we empirically investigate on CLGU dominance linked to poverty one. This is
motivated by recent theoretical works which can be used to test whether poverty or social welfare has increased or decreased over the last decades in various regions of the world, allowing for variations in populations sizes and using ranges of poverty lines and values of critical levels. Moreover, the theoretical works develop some procedure which helps us to set some relevant estimations of the ranges of the critical levels. This might serve to extend the Blackorby and Donaldson (1984)’s CLGU and complete the theoretical analysis carried out in Tranroy and Weymark (2007). The latter built a dominance criterion on the critical level principle and showed that this criterion is equivalent to the generalized Lorenz dominance criterion. Therefore, it is limited to second-order of dominance.

The remainder of the paper is structured as follows. In Section 2 we briefly summarize the statistical tests that are used for the dominance relations analysis as well as the critical levels estimations. Section 3 presents some estimations based on a few simulation. Section 4 describes the data and the methodological decisions used to set up it. It also analyses the empirical results. Section 5 concludes.

2 Statistical inference

This section sums up the approaches used to produce the estimations and the statistical tests retained to analyse dominance relations. Most of them are described in Davidson and Duclos (2000) and Davidson and Duclos (2006). We apply statistical tests for poverty orderings including the critical level.

2.1 Dominance analysis

The equivalence between poverty dominance and welfare dominance allows restricting to poverty dominance analysis. We used the FTG poverty indices (Foster, Greer, and Thorbecke 1984). For two given populations $u$ and $v$ with size $M$ and $N$ where $M$ may be lower or equal to $N$, denote their cumulative distributions $F$ and $G$ respectively. To compare the social welfare of these populations, we assume that there is a critical level for the populations. For instance, social welfare functions of $u$ and $v$ take the forms

$$W(u; \alpha) = \sum_{i=1}^{M} (g(u_i) - g(\alpha))$$

\[1\]

\[1\] See for instance Zabsonré and Duclos (2009)
and

\[ W(v; \alpha) = \sum_{j=1}^{N} (g(v_j) - g(\alpha)), \]  

(2)

where \( u_i \) refers to the income of individual \( i (i \leq M) \) in \( u \), \( v_j \) is the income of individual \( j (j \leq N) \) in \( v \) and \( g \) is some appropriate transformation. \( \alpha \) is the critical level. Hence, the social welfare of population \( u \) and that of population \( v \) remain unchanged when adding individuals with an income level equal to \( \alpha \). We now turn to poverty concern. For any poverty line \( z \), let define the poverty index of order \( s (s \geq 1) \) for the population \( u \) associated to the critical level \( \alpha \) as

\[ P_{F_\alpha}^s(z) = \int_0^z (z-u)^{s-1}dF_\alpha(u), \]  

(3)

where \( F_\alpha(z) = \frac{M}{N}F(z) + \frac{N-M}{N}I(\alpha \leq z) \). \( I(\cdot) \) is an indicator function, with value 1 if the condition is true and 0 if not. Similarly, the poverty index of the population \( v \) is defined as

\[ P_G^s(z) = \int_0^z (z-v)^{s-1}dG(v). \]  

(4)

Our main objectif in this part of the paper is to present some statistical tests that can be used to test whether a larger population dominates a smaller one at order \( s \) over an interval of poverty lines and also taking into account a range of critical levels. Two approaches can be followed to conduct this test. The first is based on the following formulation:

\[ H_{0}^s : P_{G}^s(z) - P_{F_\alpha}^s(z) \leq 0 \quad \text{for all} \quad (z, \alpha) \]  

(5)

\[ H_{1}^s : P_{G}^s(z) - P_{F_\alpha}^s(z) > 0 \quad \text{for some} \quad (z, \alpha). \]  

(6)

This kind of formulation is called “union-intersection” test. It simply amounts to define a null of dominance and an alternative of non dominance. It has been used and applied in many works where, a Wald statistic or a test statistic based on the supremum of the difference between the poverty indices\(^2\) is generally retained to

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\(^2\)The statistics tests define like that are frequently derived in the Kolmogorov-Smirnov type tests.
test the dominance. See for example Bishop, Formby, and Thistle (1992), Barrett and Donald (2003) and Lefranc and Trannoy (2006). Making use of the Wald statistic here requires to choose a fixed number of points \((z, \alpha)\) on which the hypotheses being tested. However, this procedure alters the above formulation and hence is different from it. Nevertheless, the method which relies on the supremum statistic is free from the problem related to the choice of the fixed number of points. But as Davidson and Duclos (2006) underlined it, because the two procedures (the Wald test and the supremum test) are based on the union-intersection formulation, rejecting the null of dominance and inferring non dominance may fail to rank the two populations. Indeed, it can end up to a situation of dominance in spite of the rejection of dominance. The second approach reverses the roles of (5) and (6) and posits the hypotheses as

\[
H_0^s : P_G^s(z) - P_F^s(z) \geq 0 \quad \text{for some } (z, \alpha) \quad (7)
\]
\[
H_1^s : P_G^s(z) - P_F^s(z) < 0 \quad \text{for all } (z, \alpha). \quad (8)
\]

This formulation is the type of “intersection-union” test. Contrary to the first formulation, the null is the hypothesis of non dominance and the alternative is the hypothesis of dominance. This test has been employed by Howes (1993) and Kaur, Prakasa Rao, and Singh (1994). Both computed a minimum value of the \(t\)-statistic but differently. The former used a fixed number of points while the latter considered an interval. Other statistics can be used to conduct the test \(H_0^s\) versus \(H_1^s\) in (7) and (8). Among them are especially the empirical likelihood ratio (ELR) statistics which were first proposed by Owen (1988). For a comprehensive account of EL technique and its properties, see Owen (2001). Here, we follow the procedure of Davidson and Duclos (2006) which can be also found in Batana (2008), Chen and Duclos (2008), and Davidson (2008). But at the difference of these works and as it will be more clear below, we pay attention to the sizes of the two populations in computing the two statistics and take into account the critical level. We use the ELR statistic and the \(t\)-statistic to test \(H_0^s\) versus \(H_1^s\) in the same manner as Davidson and Duclos (2006). Thus, let denote by \(m\) and \(n\) the sample sizes of populations \(u\) and \(v\) respectively and \(w_i^u\) and \(w_j^v\) the sampling weights associated to observation of individual \(i\) in the sample of \(u\) and individual \(j\) in the sample of \(v\) respectively. We denote by \(U\) the set of sample distinct values of \((u_1, u_2, \ldots, u_m)\) and \(V\) the set of sample distinct values of \((v_1, v_2, \ldots, v_n)\). We define \(c\) as the ratio \(\sum_{i=1}^{m} w_i^u / \sum_{j=1}^{n} w_j^v\) and \(p_i^u\) and \(p_j^v\), the empirical probabilities.
The ELR statistic is similar to an ordinary LR statistic and results from the maximisation of an empirical loglikelihood function (ELF) which is defined as

$$ELF(z, \alpha) = \max_{p_t^u, p_j^v} \left[ \sum_{u_i \in U} \frac{w_i^u}{\sum_{u_i \in U} w_i^u} \log p_i^u + n \sum_{v_j \in V} \frac{w_j^v}{\sum_{v_j \in V} w_j^v} \log p_j^v \right]$$

subject to

$$\sum_{u_i \in U} p_i^u = 1, \sum_{v_j \in V} p_j^v = 1$$

and

$$c \sum_{u_i \in U} p_i^u (z - u_i)_+^{s-1} + (1 - c) (z - \alpha)_+^{s-1} \leq \sum_{v_j \in V} p_j^v (z - v_j)_+^{s-1}.$$

The procedure of Davidson and Duclos (2006) computes the ELR statistic so that the null hypothesis be verified. To do so, we impose the condition (11) as it appears in the above problem of maximisation. We just need only one \((z, \alpha)\) in the interior of \([0, z^+] \times [0, \alpha^+]\) to verify (11). Then, in order to avoid the situation for the lowest values of \((z, \alpha)\) where (11) can be also hold, we restrict the \([0, z^+] \times [0, \alpha^+]\) interval to \([z^-, z^+] \times [\alpha^-, \alpha^+]\) where \(\alpha^- > 0 < z^-\). When there is non-dominance in the samples, the constraint (11) does not matter and the result of the maximisation is the same as one obtains without specifying (11). In such case, the maximised ELF does not depend on \(z\) and \(\alpha\) and we simply denote it by \(ELF\). Hence, the resulting probabilities are given by

$$p_i^u = \frac{w_i^u}{\sum_{u_i \in U} w_i^u} \quad \text{and} \quad p_j^v = \frac{w_j^v}{\sum_{v_j \in V} w_j^v}.$$  

(12)

In the case where there is dominance in the samples, the constraint (11) matters and the probabilities obtained from the resolution using Lagrangian method are:

$$p_i^u = \frac{w_i^u}{\sum_{u_i \in U} w_i^u} m - \rho \left( \nu - c (z - u_i)_+^{s-1} - (1 - c) (z - \alpha)_+^{s-1} \right),$$

(13)

and
\[ p_j^y = \frac{\sum_{v_j \in V} w_j^y n}{\sum_{v_j \in V} w_j^y n + \rho (\nu - (z - v_j)_+^{s-1})}. \]  

(14)

The constants \( \rho \) and \( \nu \) which enter in the formulas of the probabilities are the solutions of the following equations

\[
\begin{cases}
    c \sum_{u_i \in U} p_i^u (z - u_i)_+^{s-1} + (1 - c) (z - \alpha)_+^{s-1} = \sum_{v_j \in V} p_j^v (z - v_j)_+^{s-1} \\
    \sum_{v_j \in V} p_j^v (z - v_j)_+^{s-1} = \nu
\end{cases}
\]

with \( p_i^u \) and \( p_j^v \) given in (13) and (14). Of course, when \( s > 1 \), the solutions cannot be found analytically, so a numerical method must be used to solve these equations. Thus, for any pair \((z, \alpha)\), the ELR statistic is

\[
LR (z, \alpha) = 2 [ELF - ELF (z, \alpha)]
\]

(15)

\[
LR (z, \alpha) = \begin{cases} 
0 & \text{if } \hat{P}_{sG}^s (z) \geq \hat{P}_{sF}^s (z) \\
LR & \text{if not.}
\end{cases}
\]

(16)

Recall that \( \hat{P}_{sF}^s (z) \) and \( \hat{P}_{sG}^s (z) \) are respectively the sample equivalents of \( P_{sF}^s (z) \) and \( P_{sG}^s (z) \). They are given by

\[
\hat{P}_{sF}^s (z) = \frac{m}{n} \sum_{i=1}^{m} w_i^u (z - u_i)_+^{s-1} \left/ \sum_{j=1}^{n} w_j^v \right. + \left( 1 - \frac{m}{n} \sum_{i=1}^{m} w_i^u \right) \left/ \sum_{j=1}^{n} w_j^v \right. (z - \alpha)_+^{s-1}
\]

(17)

and

\[
\hat{P}_{sG}^s (z) = \sum_{j=1}^{n} w_j^v (z - v_j)_+^{s-1} \left/ \sum_{j=1}^{n} w_j^v \right.
\]

(18)

The \( t \)-statistic is the same as in Kaur, Prakasa Rao, and Singh (1994) and is simply given by the minimum of \( t(z, \alpha) \) over the \([z^-, z^+] \times [\alpha^-, \alpha^+]\) interval, and where

\[
t(z, \alpha) = \frac{\hat{P}_{sG}^s (z) - \hat{P}_{sF}^s (z)}{[\hat{\var} (\hat{P}_{sG}^s (z) - \hat{P}_{sF}^s (z))]^{1/2}}
\]

(19)
for any pair \((z, \alpha)\). Using the two statistics, we can deal either with asymptotic tests or bootstrap tests. The latter are conducted by means of the empirical probabilities obtained in the EL approach. For asymptotic tests and for a test of level \(\beta\), the decision rule is to reject \(H_0^s\) if the minimum of \(t(z, \alpha)\) over all pairs of \((z, \alpha)\) considered, exceeds the critical value associated to \(\beta\) of the standard normal distribution. The minimisation statistic is also applied on \(LR(z, \alpha)\). According to Davidson (2008), the minimum of ELR statistic and the square of the minimum of \(t\)-statistic are asymptotically equivalent. In our case, the same result holds for the minimum of \(LR(z, \alpha)\) and the square of the minimum of \(t(z, \alpha)\). See Appendix for more details. Bootstrap tests are conducted in case where there is dominance in the original samples. Following previous works, we generate 399 bootstrap samples that satisfies the null hypothesis and for each of them, we compute the two statistics as in the original data. Formally, the bootstrap method is set up in the following way.

**Step 1:** For two samples drawn from two different populations, compute \(LR(z, \alpha)\) and \(t(z, \alpha)\) for any pair \((z, \alpha)\) in \([z^-, z^+] \otimes [\alpha^-, \alpha^+]\) as described above. If there exists at least one \((z, \alpha)\) for which \(P_{G}(z) = \hat{P}_{F, \rho}(z) \geq 0\), then \(H_0^s\) cannot be rejected. Choose a value equal to 1 for the bootstrap \(P\) value and stops the bootstrap computing process here. If not, continue to the next step.

**Step 2:** Search for the minima statistics, that is to say, find the minimum of \(LR(z, \alpha)\) and the minimum of \(t(z, \alpha)\) over all pairs \((z, \alpha)\). Suppose that the minimum is obtained with \((\tilde{z}, \tilde{\alpha})\) and denote by \(\tilde{p}_{u}^{\alpha}\) and \(\tilde{p}_{v}^{\alpha}\) the resulting probabilities evaluated at \((\tilde{z}, \tilde{\alpha})\).

**Step 3:** Use \(\tilde{p}_{u}^{\alpha}\) and \(\tilde{p}_{v}^{\alpha}\) to generate the bootstrap samples by resampling the original data with these probabilities. The bootstrap samples are then drawn with unequal probabilities. Thus, it can result from it that in some bootstrap samples, the estimated size of the smaller population \(u\) becomes larger than that of the larger population \(v\). In such cases, it is important to reverse the roles of \(F_{\alpha}\) and \(G_{\alpha}\).\(^3\)

**Step 4:** Consider 399 replications and for each of them, draw bootstrap samples of size \(m\) for \(u\) and of size \(n\) for \(v\). In each replication, use the bootstrap data and taking account the remark of the previous step. Compute the two

\(^3\)This problem does not occur in the samples where all the observations have almost the same weights. In fact, in such situation, the unequal probabilities sampling is just identical to the simple random sampling.
statistics $LR_b(z, \alpha)$ and $t_b(z, \alpha)$ for any $b \leq 399$ as in the original data, but define $t_b(z, \alpha)$ as

$$t_b(z, \alpha) = \frac{\hat{P}^g_{\alpha}(z) - \hat{P}^s_{\alpha}(z)}{\sqrt{\text{var} \left( \hat{P}^s_{G, \alpha}(z) - \hat{P}^s_{F}(z) \right)}}^{1/2},$$

(20)
in case where the point in the step 3 about the estimated sizes matters. Suppose that the minimum is attained with $(\hat{z}, \hat{\alpha})$. Denote the minima statistics by $LR_b(\hat{z}, \hat{\alpha})$ and $t_b(\hat{z}, \hat{\alpha})$. If there is non dominance in the samples, put $t_b(\tilde{z}, \tilde{\alpha}) = 0$.

Step 5: Compute the $P$ values of the bootstrap statistics as the proportion of $LR_b(z, \alpha)$ that are greater than the ELR statistic obtained with the original data, $LR(\hat{z}, \hat{\alpha})$, and as the proportion of $t_b(z, \alpha)$ that are greater than $t(\hat{z}, \hat{\alpha})$, the $t$-statistic obtained with the original data.

Step 6: Reject the null of non dominance if the bootstrap $P$ values are lower than some specified nominal levels.

As we can see, the tests (7) and (8) that we described above are made over ranges of poverty lines but, also over ranges of critical levels to avoid the arbitrariness of the choice of a single poverty line or a single critical level.

2.2 Critical-level estimation

Given the lack of empirical or ethical consensus on the value of the critical-level, we seek evidence on the ranges of critical-levels that can order distributions (Blackorby, Bossert, and Donaldson 1996, Trannoy and Weymark 2007). We follow Davidson and Duclos (2000) (henceforth DD) for the derivation of the asymptotic properties of estimators of $\alpha_e$ and $\alpha^e$ defined in the previous paragraph. We focus on a situation in which we have two samples of income from two populations of possibly different sizes. For the purpose of the estimations, we assume that data have been generated by a data generating process (DGP) from which a finite (but usually large) population is generated. We sometimes need to assume that this DGP is continuous, but this is different from saying that the populations must be continuous (or of infinite size) too. For that, let consider again the two populations $u$ and $v$ of sizes $M$ and $N$ respectively. Suppose that their are independent and their moments of order $2(s - 1)$ are finite. We define $F$ and $G$ as the
distribution functions of the DGP that generates \( u \) and \( v \). Suppose that we have two samples drawn from \( u \) and \( v \) and assume for simplicity that they are independent. Denote by \( m \) and \( n \) the sizes of the two samples. Notice that we define \( \alpha_s \) and \( \alpha^s \) respectively as follows:

\[
\alpha_s = \max\{\alpha | P^s_{u_\alpha}(z) \geq P^s_v(z) \text{ for all } z \leq z^+\} \tag{21}
\]

and

\[
\alpha^s = \min\{\alpha | P^s_{u_\alpha}(z) \leq P^s_v(z) \text{ for all } z \leq z^+\}. \tag{22}
\]

In order to have poverty comparisons made robustly over ranges of poverty lines, we also define critical values for the maximum poverty line as:

\[
z^+_s = \max\{z^+ | P^s_{u_\alpha}(z) \geq P^s_v(z) \text{ for all } z \leq z^+\} \tag{23}
\]

and

\[
z^{s+} = \max\{z^+ | P^s_{u_\alpha}(z) \leq P^s_v(z) \text{ for all } z \leq z^+\} \tag{24}
\]

where \( \alpha \) is a fixed critical level. As we see, \( z^+_s \) is the maximum poverty line for which \( v \) dominates \( u \) and \( z^{s+} \) is the maximum poverty line for which \( u \) dominates \( v \). Given the definitions (21) and (22) and assuming that \( \alpha_s \) and \( \alpha^s \) exist, one can notice that the best situation to estimate \( \alpha_s \) is that where there is a clear dominance of \( v \) over \( u \). In the same way the best situation to estimate \( \alpha^s \) is that where there is dominance of \( u \) over \( v \). Therefore, that is why we posit the following assumptions. For \( \alpha_s \), we suppose that

\[
\begin{cases}
\frac{M}{N} P^s_F(z) \geq P^s_G(z) \text{ for all } z \leq \alpha_s \\
\frac{M}{N} P^s_F(z) < P^s_G(z) \text{ for some } z \geq \alpha_s + \epsilon \text{ and } z \leq z^+ \tag{VDU_s}
\end{cases}
\]

where \( \epsilon \) is some arbitrarily small positive value. For \( \alpha^s \), we consider first the case of \( s = 1 \) and we suppose that

\[
\begin{cases}
\frac{M}{N} P^1_F(z) + \frac{(N-M)}{N} I(\alpha^1 \leq z) \leq P^1_G(z) \text{ for all } z \leq z^+ \\
\frac{M}{N} P^1_F(z) + \frac{(N-M)}{N} > P^1_G(z) \text{ for some } z \leq \alpha^1 + \epsilon \tag{UDV_1}
\end{cases}
\]

where \( \epsilon \) is again some arbitrarily small positive value. When \( s \geq 2 \), we slightly modify the above assumptions and state as this
\[
\begin{cases}
\frac{M}{N} P_s^u(z) + \frac{(N-M)}{N} (z - \alpha_s)^{s-1}_+ \leq P_s^v(z) & \text{for all } z \leq z^+ \\
\frac{M}{N} P_s^v(z^s) + \frac{N-M}{N} (z^s - \alpha_s)^{s-1}_+ = P_s^v(z^s) & \text{for } \alpha_s < z^s \leq z^+
\end{cases}
\]

(UDV_s)

with \((z - x)^{s-1}_+ = \max[(z - x)^{s-1}, 0]\). The assumptions VDU_s and UDV_s are important for the estimation of \(\alpha_s\) and \(\alpha^s\). In order to better understanding their role in the estimation of \(\alpha_s\) and \(\alpha^s\) of the two populations \(u\) and \(v\), let consider the case where \(s = 1\), and let illustrate graphically what happens with the dominance involving \(\alpha_1\) and \(\alpha^1\). The following figures graph the cumulative distributions functions for different values of \(\alpha_1\).

Figure 1: Poverty incidence curves with a zero value of the lower bound of the critical-levels

In Figure [2], we suppose that the larger population \(v\) dominates the lower population \(u\) for a certain range of the poverty line. This is expressed by the fact that the poverty curve \(G_v\) of \(v\) is under the poverty curve \(F_u\) of \(u\) up to \(\alpha_1 > 0\). At the value \(\alpha_1\), the two functions cross, but \(v\) still dominates \(u\) when the critical level is equal to \(\alpha_1\). However, \(v\) may not dominate \(u\) when the critical level has a certain value \(\alpha_0 > \alpha_1\). In Figure [3], \(u\) is assuming to dominate \(v\). The dominance of \(u\)
over \( v \) is preserved when the critical level has a value at least equal to \( \alpha^1 \). But it cannot still hold for any critical level lower \( \alpha_0 \) than \( \alpha^1 \).

In the case where \( \text{VDU}_s \) is verified, it is possible that \( \alpha^s \) cannot be estimated. On the other hand, if \( \text{UDV}_s \) holds, it is possible that \( \alpha_s \) would not be estimated. This is because the assumptions \( \text{VDU}_s \) and \( \text{UDV}_s \) go in opposite directions. However, there can be a particular situation where the two critical values may be estimated at the same time. In such case, it is not difficult to see that \( \alpha_s \leq \alpha^s \) as it is shown in Figure 4. But this situation is essentially the one where the dominance of \( u \) over \( v \) is weak enough. In other words, the two distributions \( \frac{M}{N} F \) and \( G \) may coincide over a \([0, \bar{z}]\) interval with \( \bar{z} < 1^4 \).

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4In empirical analysis, it would be difficult to find distributions that verify such a relation.
Figure 3: Poverty incidence curves and the upper bound of the critical-levels

Figure 4: Critical-levels and dominance
3 A few simulation

As in [Davidson and Duclos (2006)] and [Davidson (2008)] and without loss of generality, we define the distributions $F$ and $G$ over the $[0, 1]$ interval, as higher-order of dominance is invariant to increasing affine transformations. Let population $v$ have an uniform distribution on $[0, 1]$ and population $u$ being piecewise linear distributed, that is to say uniform on 20 equal segments belonging to the $[0, 1]$ interval. The upper limits of these segments are 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, and 1.00. Because $v$ is supposed to have an uniform distribution, these upper limits correspond to the cumulative probabilities for $v$ at these points. In the first case, the cumulative probabilities for $u$ at the upper limit of each segment are respectively 0.15, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.82, 0.85, 0.87, 0.90, 0.95, 0.97 and 1.00. Clearly, $v$ dominates $u$. As mentioned in the previous section, this is the best situation to find $\alpha_s$. For instance, in Figures[5] and [6] we see that $\alpha_1 = 0.3$ and $\alpha_2 = 0.6$. Then, population $v$, which is the larger population, dominates population $u$, the smaller one, at first order, given any critical level at most equal to 0.3. And the second order is obtained with $\alpha \leq 0.6$.

In the second case, we let the population $u$ dominates the population $v$. Thus, we choose the cumulative probabilities for $u$ as 0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.45, 0.55, 0.65, 0.70, 0.75, 0.80 and 1.00. We can then find the critical levels $\alpha^s$. Figures[7] and [8] show that $\alpha^1 = 0.4$ and $\alpha^2 = 0.2$. Hence, the smaller population $u$ dominates the larger one, at first order, for any critical level $\alpha \geq 0.4$. And the second order is obtained with $\alpha \geq 0.2$. Notice that all the experiments are based on samples sizes $m = 1000$ for $u$ and $n = 1500$ for $v$.

The first feature we can notice when looking at Table[1] and Table[2] is the relation between the critical levels and the order of dominance $s$. As mentioned in the previous section, the estimations show that $\alpha_s$ is increasing with $s$ and $\alpha^s$ is decreasing with $s$. However it makes no sense if one wants to compare them, because strictly speaking, they are estimated using two different pairs of populations. The second feature suggests that there should be a relation between critical levels and populations sizes.

Let consider a simple configuration in which the two distributions did not change in spite of the increase of the ratio $M/N$. The simulation results presented in Table[1] and Table[2] reveal that the critical levels may be sensitive to a change in sizes. This would be closely akin to the suggestion made by [Blackorby, Bossert, and Donaldson (2000)] that the critical level may be number dependent. Indeed,
Figure 5: Population poverty incidence curves and the lower bounds of the critical levels

\[ F_{u_{\alpha_1}}(z) \]
\[ \frac{M}{N} F_u(z) \]
\[ G_v(z) \]

\[ \alpha_1 = 0.3 \]
Figure 6: Population poverty gap curves and the lower bounds of the critical levels

\[ P_{u_{\alpha_2}}^2(z) \]

\[ \frac{M}{N} P_u^2(z) \]

\[ P_{v}^2(z) \]

\[ 0 \leq \alpha_2 = 0.6 \leq z \]
Figure 7: Population poverty incidence curves and the upper bounds of the critical levels

\[ F_{u_{n1}}(z) \]
\[ \frac{M}{N} F_u(z) \]
\[ G_v(z) \]
Figure 8: Population poverty gap curves and the upper bounds of the critical levels
Table 1: Populations sizes and lower bounds of the critical level

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{M}{N} = \frac{1}{4}$</th>
<th>$\frac{M}{N} = \frac{1}{2}$</th>
<th>$\frac{M}{N} = \frac{2}{3}$</th>
<th>$\frac{M}{N} = \frac{3}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>0.05</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>0.05</td>
<td>0.3</td>
<td>0.6</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Table 2: Populations sizes and upper bounds of the critical level

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{M}{N} = \frac{1}{4}$</th>
<th>$\frac{M}{N} = \frac{1}{2}$</th>
<th>$\frac{M}{N} = \frac{2}{3}$</th>
<th>$\frac{M}{N} = \frac{3}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>0.95</td>
<td>0.85</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>0.5</td>
<td>0.35</td>
<td>0.2</td>
<td>0.15</td>
</tr>
</tbody>
</table>

these tables show how the critical-levels change with the variation of the ratio $\frac{M}{N}$. As we can see, as soon as the difference between the sizes decreases, the value of the lower bound increases whereas the value of the upper bound decreases. And if the difference between the sizes of populations is sufficiently high, the value of the lower bound appears to be very small and that of the upper bound to be high. This stems from the fact that when adding more individuals with the same value to the small population, the larger population will continue to dominate the small one if the value is low. Otherwise, it would not be sure that the dominance relation should hold if the value was relatively high. On the other hand, the value of the upper bound becomes low as the difference between the sizes becomes increasingly little. Recall that all these results are based on the dominance relations set above, but after some changes made on the distributions, we find that the variations have not changed.

4 Illustration using Canadian data

The data are drawn from the Canadian Surveys of Consumer Finances (SCF) for 1976 and 1986, and the Canadian Survey of Labour and Income Dynamics (SLID) for 2006. We use per capita net income as a measure of individual well-being and
not consider the multiples aspects around it. To compute per capita net income, we use the equivalence scale often employed by Statistics Canada. This equivalence scale relies on a factor of 1 for the oldest person in the family, 0.4 for the second oldest person and all other members aged at least 16 and 0.3 for the remaining members under age 16. In order to take into account the differences in temporal and regional or spatial prices, we adjust incomes by using the market basket measure which seems to be the best way to consider the time and space differences. We also use the procedure based on the consumer price index to convert data into 2002 constant dollars. The sample sizes from 1976, 1986 and from 2006 are respectively 28,613, 36,389 and 28,524. When applying the sampling weights to observations, we estimate Canadian populations sizes for 1976 at 22,230,280, for 1986 at 25,384,476 and for 2006 at 31,853,378. Comparisons are made using two populations, a small one and a large one. Naturally, we consider the populations of 1976 and 1986 being the small populations and the population of 2006 being the large one. Notice that we affect the value of 0 to all negative incomes. This concerns only 0.18% of the observations for the sample of 1976. For the sample of 1986, the proportion is 0.15% and 0.13% for the sample of 2006. For the poverty dominance analysis, we set the upper bound of the poverty line \( z^+ \) at $20,000 and most of the estimations we made rely on the range \([0, 20,000]\), expecting that this range contains the appropriate poverty line\(^5\). However, for other estimations, we sometimes set \( z^+ \) at an arbitrary high value. Table 3 presents the results for the restricted dominance tests based on the range of poverty lines \([9,500, 20,000]\) and the range of the critical levels \([5,000, 15,000]\) and where ND stands for non-dominance.

<table>
<thead>
<tr>
<th>Dominance tests</th>
<th>Order ( s )</th>
<th>Asymptotic ( p )-value</th>
<th>Bootstrap ( p )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1976 versus 2006</td>
<td>( s = 1 )</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1986 versus 2006</td>
<td>( s = 1 )</td>
<td>0.000</td>
<td>0.027</td>
</tr>
<tr>
<td>1976 versus 2006</td>
<td>( s = 2, 3 )</td>
<td>ND</td>
<td>ND</td>
</tr>
<tr>
<td>1986 versus 2006</td>
<td>( s = 2, 3 )</td>
<td>ND</td>
<td>ND</td>
</tr>
</tbody>
</table>

\(^5\)In their poverty analysis applied to Canada, \[\text{Chen and Duclos (2008)}\] used the same value for the upper bound of the poverty line.
Once the test reveals a dominance of 2006 over 1976 and 1986 at the first-order, we do not need to continue the test for a higher-order of dominance, since it is satisfied. Then we only give results involving first-order of dominance for asymptotic tests. At 5% significance level, the population of 2006 dominates the populations of 1976 and 1986 at first-order, when the test is an asymptotic test. But for bootstrap test, we do not found clearly a dominance of 2006 over 1976 and 1986, even if we obtain a too low value of the p-value. The reason is that the bootstrap frequently generates a pair of samples without dominance. In this case, the bootstrap statistic cannot be higher than the statistic obtained with the original samples. But in the situation where $\alpha^+ \leq z^-$, we find that the population of 2006 dominates the populations of 1976 and 1986 at first-order, whether we use an asymptotic test or a bootstrap test. This result is trivial. Notice that the dominance relations remain unchanged when the lower bound of the critical level is arbitrary close to 0. This is a result already given in Section 2. When the larger population dominates the smaller one for a given value $\alpha$ of the critical level, then the larger population continues to dominate the smaller one for any value of the critical level lower than $\alpha$.

Table 4: Estimates of the critical level

<table>
<thead>
<tr>
<th>$s$</th>
<th>1976 versus 2006</th>
<th>1986 versus 2006</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\alpha}_s$</td>
<td>$\hat{\sigma}_s$</td>
</tr>
<tr>
<td>$s = 1$</td>
<td>50,845.19</td>
<td>878.49</td>
</tr>
<tr>
<td>$s = 2$</td>
<td>75,149.54</td>
<td>2,208.98</td>
</tr>
<tr>
<td>$s = 3$</td>
<td>103,403.25</td>
<td>3,545.71</td>
</tr>
<tr>
<td>$s = 4$</td>
<td>133,041.923</td>
<td>5,015.45</td>
</tr>
</tbody>
</table>

Notes: All amounts are in 2002 constant dollars.

Table 4 gives the estimates of the critical levels for which the population of 2006 dominates the populations of 1976 and 1986 respectively. It is easy to see that the dominance of 2006 over 1976 is more clear than that of 2006 over 1986. Indeed, the estimates of $\alpha_s$ are very high when the smaller population is that of 1976 compared to the values of $\alpha_s$ obtained with the population of 1986. For instance, any critical level lower or equal to $50,845.19$ leads to the dominance of 2006 over 1976 at first-order. However, the dominance of 2006 over 1986 requires small values of $\alpha_s$. In the case of first-order dominance, it only amounts
to $465. This suggests that for some values of $s$ slightly above those related to 1986 versus 2006 at the right hand side in Table 4, the population of 2006 should not dominate the population of 1986 in poverty and hence in welfare. Remark that for any critical level $\alpha$ belonging to $[80, 80,000]$ and for $s \geq 1$, $z^*_{s+}$ is arbitrary high when the dominance concerns 1976 and 2006. This is again consistent with the result mentioned above that the dominance of 2006 over 1976 is more obvious.

Notice that we are not able to estimate the critical level $\alpha^*$ and the critical value $z^*_{s+}$ because the assumptions UDV fail to be satisfied. This is due to the fact that there are more individuals with equivalent income equal to 0 in the samples of 1976 and 1986 than in the sample of 2006. Consequently, the estimated number of poor in the populations of 1976 and 1986 exceeds that in the population of 2006 and it becomes difficult to obtain the dominance of 1976 and 1986 over 2006.

5 Conclusion

In this paper, we extend the approach of the critical-level generalized utilitarianism of Blackorby and Donaldson (1984) by testing dominance and estimating the critical levels. We apply the results using Canadian data. In particular, we compare welfare in Canada between two distant dates viz., a period of time at least equal to two decades. On the one hand, the comparison concerns 1976 and 2006, and on the other hand, it concerns 1986 and 2006. We use asymptotic and bootstrap tests to test for dominance involving some possible values of the critical level. We find that the recent year 2006 dominates the earlier years 1976 and 1986. Therefore, this suggests that there has been a welfare improvement for Canadian population during the last decades. This may be due to the periods of recession of the 70s years and the 80s years. We also deal with some simulations that show that the sizes of populations may be a great concern for welfare ranking.
Appendix

Proposition 1
For $s \geq 1$, suppose that the ratio $r = \frac{m}{n}$ remains constant as $m$ and $n$ tend to infinity. Then, for any pair $(z, \alpha)$ in the interior of $[z^-, z^+] \otimes [\alpha^-, \alpha^+]$, such that $P^s_G(z) = P^s_{F_\alpha}(z)$, the statistic $LR(z, \alpha)$ tends to the square of the asymptotic $t$-statistic where

$$t^2(z, \alpha) = \frac{(\Delta P^s(z, \alpha))^2}{\text{Avar}\left(\sqrt{n} \left(\hat{P}^s_G(z) - \hat{P}^s_{F_\alpha}(z)\right)\right)}$$

where $\Delta P^s(z, \alpha) = p \lim_{m,n \to \infty} \sqrt{n} \left(\hat{P}^s_G(z) - \hat{P}^s_{F_\alpha}(z)\right) = O(1)$

Proof
Recall that $\frac{1}{2} ELR = [ELF - ELF(z, \alpha)]$. To simplify the notations, let denote $a_i = \frac{mw^u_i}{\sum_{i \in U} w^u_i}$, $b_j = \frac{nw^v_j}{\sum_{j \in V} w^v_j}$ and $c = \sum_{i=1}^{m} w^u_i / \sum_{j=1}^{n} w^v_j$. We have that

$$ELF = \sum_{u_i \in U} a_i \log\left(\frac{a_i}{m}\right) + \sum_{v_j \in V} b_j \log\left(\frac{b_j}{n}\right).$$

(26)

For ease of exposition, we denote $\theta = (1 - c)(z - \alpha)^{s-1}_+, U_i = (z - u_i)^{s-1}_+$ and $V_j = (z - v_j)^{s-1}_+$. Using the results of the empirical probabilities obtained in the resolution of (9), (10) and (11), we have

$$ELF(z, \alpha) = \sum_{u_i \in U} a_i \log\left(\frac{a_i}{m - \rho(\nu - c U_i - \theta)}\right) + \sum_{v_j \in V} b_j \log\left(\frac{b_j}{n + \rho(\nu - V_j)}\right).$$

(27)

Hence,

$$\frac{1}{2} ELR = \sum_{u_i \in U} a_i \log\left(1 - \frac{\rho(\nu - c U_i - \theta)}{m}\right) + \sum_{v_j \in V} b_j \log\left(1 + \frac{\rho(\nu - V_j)}{n}\right).$$

Using Taylor’s expansion applied on the log function and the condition that $P^s_G(z) = P^s_{F_\alpha}(z)$, $\frac{1}{2} ELR$ becomes
\[ \frac{1}{2} ELR = \frac{\rho^2}{2} \left[ \frac{\nu^2}{m} - \frac{1}{m} \sum_{u_i \in U} \frac{w_i^u}{u^u} (cU_i + \theta)^2 + \frac{\nu^2}{n} - \frac{1}{n} \sum_{v_j \in V} \frac{w_j^v}{w^v} V_j^2 \right] . \]

In order to complete the expression of the ELR, we must replace the Lagrange multipliers by their values. Because (11) gives that

\[ c \sum_{u_i \in U} p^u_i (z - u_i)^{s-1} + (1 - c) (z - \alpha)^{s-1} = \sum_{v_j \in V} p^x_j (z - v_j)^{s-1} \quad (28) \]

and this is exactly \( \nu \), then using again Taylor’s expansion and the expressions of \( p^u_i \) and \( p^x_j \), we solve for \( \rho \) and find that

\[ \rho = \frac{\hat{P}_G^s(z) - \hat{P}_{F_0}^s(z)}{\left[ \frac{\nu^2}{m} - \frac{1}{m} \sum_{u_i \in U} \frac{w_i^u}{u^u} (cU_i + \theta)^2 + \frac{\nu^2}{n} - \frac{1}{n} \sum_{v_j \in V} \frac{w_j^v}{w^v} V_j^2 \right] } . \]

Therefore,

\[ ELR = \frac{n \left( \hat{P}_G^s(z) - \hat{P}_{F_0}^s(z) \right)^2}{\left[ \frac{1}{r} \hat{P}_{F_0}^s(z)^2 - \frac{1}{r} \sum_{u_i \in U} \frac{w_i^u}{u^u} (cU_i + \theta)^2 + \hat{P}_G^s(z)^2 - \sum_{v_j \in V} \frac{w_j^v}{w^v} V_j^2 \right] } . \]

It is not difficult to show that the denominator of ELR coincides with that of \( t^2(z, \alpha) \). \( \diamond \)
References


