Conditions Ensuring the Separability of Asset Demand for All Risk Averse Investors

by

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Abstract

We explore how the demand for a risky asset can be separated into a mean effect and a hedging effect by all risk averse investors. This question has been shown to be complex out of the mean-variance framework. We restrict dependence between returns on the risky assets to regression dependence and find that the demand for one risky asset can be decomposed into a mean part based on the risk premium offered by the asset and a hedging part used against the fluctuations in the return on the other risky asset. We show that the class of two-fund separating distributions implies regression dependence. This result indicates that the class of regression dependent distributions is larger than that of two-fund separating distributions and opens the search for more general distributional hypotheses applicable to asset pricing models. Examples are discussed.
Conditions Ensuring the Separability of Asset Demand for All Risk Averse Investors

The mean-variance model of portfolio choice, as initially introduced by Markowitz (1952), has been used extensively to answer the following question: Under what conditions can the demand for one risky asset be decomposed into a mean part based on the risk premium offered by the asset and a hedging part used against the fluctuations of the other risky asset return? (Mossin (1973) and Huang and Litzenberger (1988)). Though commonly used, the mean-variance model puts strong conditions either on preferences or on return distributions (i.e. quadratic utility function or normal distribution, for example). The normal distribution has been challenged by many empirical studies (Fama (1965) and Zhou (1993)) and the quadratic utility function displays increasing absolute risk aversion (Arrow (1971)). More recently, Beaulieu, Dufour and Khalaf (2003) showed that mean-variance efficiency is still rejected (but less frequently) when non-normal distributions are considered. They concluded that more research is needed to better identify the necessary and sufficient distributional hypotheses applicable to asset pricing models.

Merton (1971) characterizes optimal dynamic portfolio strategies and shows that time-varying investment opportunities result in an optimal portfolio with two parts: a mean-variance part and an intertemporal hedging part. Merton’s result was obtained under the assumption of a Markovian diffusion process and was extended recently by Kramkov and Schachermayer (1999) to more general semi-martingales. These authors were, however, not able to find the qualitative result dissociating the hedging component from the risk premium component, for all risk averse investors.
The main objective of this article is to provide conditions ensuring the separability of asset demand for all risk averse investors. It also proposes a class of distribution functions that includes that of the two-fund separation.

We introduce and describe a new form of risk dependence, namely regression dependence. This concept is due to Tukey (1958) and was extended to quadrant dependence by Lehmann (1966). This form of non-linear dependence tells how two random variables behave together when they are simultaneously small (or large). Regression dependence is useful for portfolio management since it can take into account the simultaneous downside (upside) evolution of assets prices by introducing a natural hedging property. We shall show how regression dependence permits the decomposition of asset demand.

The two-fund separation theorem is limited to a few classes of return distributions: multivariate normal distribution (Ross (1978)), elliptical distributions (Owen and Rabinovitch (1983) and Chamberlin (1983)), and linear conditional expectation distributions (Wei, Lee and Lee (1999)). Moreover, these distributions imply separation but the converse may not be true. We shall show that the Ross mutual fund separation theorem implies the family of regression dependent distributions. We shall also provide an example of a joint distribution in the regression dependent family that is not a two-fund separating distribution. These results mean that the class of regression dependent distributions is larger than that of two-fund separating distributions.

Section I presents our model of portfolio choice and introduces the concept of regression dependence (Tukey (1958)). In this section, we also derive our main results related to the decomposition of portfolio weights into a mean part and a hedging part. In Section II, we
establish formally the links between regression dependence and mutual fund separation (see Elton and Gruber (2000) for a recent article on two-fund separation). We also provide examples of regression dependent distributions for applications in finance. Section III concludes the article.

I. Characterizing optimal portfolios

In the basic problem of portfolio choice, where a risk averse investor is allocating his wealth between a risk-free asset and a risky asset, it is well known that the optimal position to take on the risky asset (long vs. short) is a function of its risk premium. When we consider a potential portfolio of two risky assets and a risk-free asset, the optimal position to take on one risky asset is no longer solely a function of its expected excess return, but it also depends on the correlation between the risky assets, which is the hedging effect. If, for example, the two risky assets are negatively correlated, it may be optimal to hold both in positive quantities, even if one risky asset offers no risk premium.

We consider a risk averse agent who allocates his wealth (normalized to one) between one risk-free asset (with return $x_0$) and two risky assets with returns $x_i$, for $i = 1,2$. We denote the joint distribution function as $dF(x_1, x_2)dG(x_2)$, where $F$ is the conditional distribution function of $x_1$ given $x_2 = x_2$, and $G$ is the distribution function of $x_2$. We also note $[x_1, x_1]$ and $[x_2, x_2]$ as the supports for $x_1$ and $x_2$ respectively, and $\alpha_i, i = 0,1,2$, as the investment in asset $i$ chosen so as to maximize expected utility in a world where short selling is not limited and under the constraint that $\alpha_0 + \alpha_1 + \alpha_2 = 1$. The agent's end-of-period wealth $W$ is then equal to
\[ W(\alpha_1, \alpha_2) = 1 + x_0 + \alpha_1 (\bar{x}_1 - x_0) + \alpha_2 (\bar{x}_2 - x_0). \]

We define \( E \) as the expectation operator and \( m_i \) as the risk premium associated with asset \( i \), that is \( m_i = E(\bar{x}_i) - x_0 \), for \( i = 1,2 \).

The optimal portfolio solves the following program (P):

\[
\max_{\alpha_1, \alpha_2} E(u(W(\alpha_1, \alpha_2))).
\]

\( u(.) \) is the von-Neumann-Morgenstern utility function assumed to be strictly increasing, strictly concave in final wealth, and continuously differentiable to the second order.

The first-order conditions associated with an interior solution of the expected utility maximization problem are:

\[
\frac{\partial}{\partial \alpha_1} E (u(W(\alpha_1, \alpha_2))) = E ((\bar{x}_1 - x_0)u'(W(\alpha_1, \alpha_2))) = 0 \tag{1}
\]

and

\[
\frac{\partial}{\partial \alpha_2} E (u(W(\alpha_1, \alpha_2))) = E ((\bar{x}_2 - x_0)u'(W(\alpha_1, \alpha_2))) = 0. \tag{2}
\]
Since $u$ is strictly concave, the first-order necessary conditions are also sufficient for an optimal solution. In the case of independence between the risky assets, condition (1) evaluated at $\alpha_1 = 0$ can be written as

$$(E(x_1) - x_0) E(u'(1 + x_0 + \alpha_2(x_2 - x_0))),$$

which has the sign of the risk premium associated with $\tilde{x}_1$. It follows that $\alpha_1^*$ is positive if and only if $m_1$ is positive, that is if, and only if, $\tilde{x}_1$ offers a positive risk premium. The same logic applies for $\alpha_2^*$.

Allowing for dependence between risky assets returns makes the characterization of the optimal portfolio more difficult. Our task of characterizing the optimal portfolio is even more complex in a context where risk aversion is the only restriction imposed on preferences.

As an illustration of this difficulty, we consider, for a moment, the case of mean-variance preferences; precisely, we suppose $u(W) = W - \frac{b}{2}W^2$, where $b$ is a positive parameter that captures the agent risk aversion. We also assume the following regularity condition on the first derivative $u'(W) = 1 - bW > 0$ for all $W$. The explicit solution to the maximization problem (P) yields:

$$\alpha_1^* = \frac{1 - b(1 + x_0)}{b} \frac{m_1\sigma_{22} - m_2\sigma_{12}}{\Delta},$$
\[
\alpha_2^* = \frac{1 - b(1 + x_0)}{b} \frac{m_2 \sigma_{11} - m_1 \sigma_{12}}{\Delta}.
\]

where \( \sigma_{ij} = \text{Cov}(\tilde{x}_i, \tilde{x}_j) \), \( 1 - b(1 + x_0) > 0 \) from the regularity condition and
\[
\Delta = m_2^2 \sigma_{11} + m_1^2 \sigma_{22} - 2m_1m_2 \sigma_{12} + \sigma_{11} \sigma_{22} - \sigma_{12}^2 > 0
\]
from the second-order condition. It is easily observed that \( \alpha_2^* \), the optimal investment in asset 2, is a function of \( m_2 \) and of \( \sigma_{12} \).

\( \alpha_2^* \) can be decomposed into
\[
\alpha_{2h}^* = -\frac{\sigma_{12}}{\sigma_{22}} \alpha_1^* = \frac{1 - b(1 + x_0)}{b} \frac{\sigma_{12}}{\sigma_{22}} \frac{m_1 \sigma_{22} - m_2 \sigma_{12}}{\Delta},
\]
the hedging part, and
\[
\alpha_{2m}^* = \frac{1 - b(1 + x_0)}{b} \frac{1}{\sigma_{22}} \frac{\sigma_{11} \sigma_{22} - (\sigma_{12})^2}{\Delta} - m_2 = km_2,
\]
the mean part. Since \( k \) is strictly positive, \( \alpha_{2m}^* \) is proportional to \( m_2 \) and
\[
\text{Sign} \left( \alpha_1^* \alpha_{2h}^* \right) = -\text{Sign}(\sigma_{12}).
\]

We now introduce the next definition that will characterize the set of return distributions in this article.
**Definition 1** (Tukey, 1958): The distribution of $\tilde{x}_1$ given $\tilde{x}_2$ shows respectively complete negative regression or complete positive regression on $\tilde{x}_2$ if the cumulative conditional distribution function $F(x_1 / x_2)$ satisfies

$$F\left(\frac{x_1}{x_2}\right) \leq F\left(\frac{x_1}{x_2}\right) \quad \text{(respectively } F\left(\frac{x_1}{x_2}\right) \geq F\left(\frac{x_1}{x_2}\right)) \quad \text{for all } x_2 \leq x_2$$

provided the equality does not always hold.

Positive regression dependence describes pairs of variables $(\tilde{x}_1, \tilde{x}_2)$ where large values of $\tilde{x}_2$ tend to be associated with large values of $\tilde{x}_1$ and small values of $\tilde{x}_2$ with small values of $\tilde{x}_1$. The opposite is true for negative regression dependence. One important property of regression dependence is that if $\tilde{x}_1$ shows complete positive (negative) regression on $\tilde{x}_2$ then the covariance between $\tilde{x}_1$ and $\tilde{x}_2$ is positive (negative). However, the converse is not true (Tong, 1980).

Another property to notice is that if $\tilde{x}_1$ shows complete negative regression on $\tilde{x}_2$ and $\tilde{x}_{11}$ is an order statistics from a random sample of $\tilde{x}_1$, then $\tilde{x}_{11}$ will show complete negative regression on $\tilde{x}_2$. A well-known particular case of regression dependence is obtained by requiring that the conditional density of $\tilde{x}_1$ given $\tilde{x}_2$ has a monotone likelihood ratio. We then say that $F(x_1 / x_2)$ shows a positive (negative) likelihood ratio dependence (Lehmann (1966)). The bivariate normal density is an example with monotone likelihood ratio dependence.

**Example 1.** Let $\tilde{x}_1 = a + b\tilde{x}_2 + \tilde{e}$, where $\tilde{x}_2$ and $\tilde{e}$ are independent. Then $\tilde{x}_1$ is positively or negatively regression dependent on $\tilde{x}_2$ as $b \geq 0$ or $b \leq 0$. In fact, the conditional distribution of $\tilde{x}_1$
given $\tilde{x}_2 = x_2$ is that of $a + bx_2 + \varepsilon$ and hence is stochastically increasing in $x_2$ if $b \geq 0$. In particular, the components of a bivariate normal distribution are positively or negatively regression dependent according to whether the correlation coefficient is positive or negative (Lehmann (1966)).

Under the assumption of regression dependence, we are able to establish our main result:

**Proposition 1:** Let $\tilde{x}_1$ show either complete negative regression or complete positive regression on $\tilde{x}_2$ and let $(\alpha_1^*, \alpha_2^*)$ be the optimal portfolio, then $\alpha_2^*$ can be decomposed as $\alpha_2^* = \alpha_{2m}^* + \alpha_{2h}^*$ with

a) $\alpha_{2m}^* \geq 0$ if and only if $E(\tilde{x}_2) \geq x_0$, and

b) $\text{Sign}(\alpha_1^* \alpha_{2h}^*) = -\text{Sign}(\text{Cov}(\tilde{x}_1, \tilde{x}_2))$,

for all risk averse investors.

**Proof:** See the Appendix.

$\alpha_{2m}^*$ and $\alpha_{2h}^*$ designate respectively the mean (or risk premium) part and the hedging part of asset 2 demand. The mean term depends on the risk premium offered by the risky asset, and the hedging term is a function of the fluctuations in the return on the other risky asset. The intuition behind Proposition 1 is natural and a significant implication of the proposition is that we need
only know the sign of the covariance to sign the hedging effect, even if we do not restrict our
analysis to the mean-variance model.

One corollary from Proposition 1 is that the net investment positions (long vs. short) on the
mean component \((\alpha^*_{2m})\) and the hedging component \((\alpha^*_{2h})\) depend solely on the distributions of
the risky assets for all risk averse investors. Preferences determine the trade-off between the risk
premium effect and the hedging effect and set the total investment of the risky asset. The result
of Proposition 1 is related to that of Ross (1978) who presented separation conditions that allow
for the optimal portfolio to exhibit two-fund separation for all risk-averse investors. In Example
2 below, we show how two-fund separating distributions are related to regression dependent
distributions.

Our next result is the following:

**Proposition 2:** Let \(\tilde{x}_1\) show either complete negative regression or complete positive regression
on \(\tilde{x}_2\) then:

\[
a) \quad \alpha^*_{2h} = 0 \quad \text{if and only if} \quad \text{Cov}(\tilde{x}_1, \tilde{x}_2) = 0, \quad \text{and}
\]

\[
b) \quad \alpha^*_{2m} = 0 \quad \text{if and only if} \quad E(\tilde{x}_2) - x_0 = 0.
\]

**Proof of Proposition 2:**
We know that if \( \operatorname{Cov}(x_1, x_2) = 0 \) then \( \alpha_{2h}^* = 0 \). It remains to show that if \( \alpha_{2h}^* = 0 \) then the two random variables have a nil covariance. Integrating by parts the left-hand-side term in (A2) yields

\[
- \int_{x_2}^{x_1} \left( \int t - E(\tilde{x}_2) / dG(t) \right) \left( \int u'(W(\alpha_1^*, 0)) \frac{\partial}{\partial x_2} dF(x_1 / x_2) \right) dx_2 = 0.
\] (4)

Under our assumption of regression dependence and since \( \int_{x_1}^{x_2} (t - E(\tilde{x}_2)) dG(t) \leq 0 \) for all \( x_2 \), in order for equality in (4) to hold, we need to have

\[
\frac{\partial}{\partial x_2} F(x_1 / x_2) = 0 \quad \text{for all} \quad x_2,
\] (5)

which means that \( \tilde{x}_1 \) and \( \tilde{x}_2 \) have a nil covariance.

Part b) of the proposition follows from Proposition 1. \( Q.E.D. \)

Asset \( \tilde{x}_2 \) can be interpreted as a derivative and Proposition 2 shows that a risk averse investor can invest money in a risky asset even though there is no risk premium associated with this risky investment. Financial risks, as opposed to insurable risks, cannot be eliminated through pooling. Financial risk reduction can be achieved by investing in a security correlated with a risky asset. The returns on this security can be highly positively correlated or highly negatively correlated.
with the basic asset. In each case, it is possible to obtain risk reduction by taking an appropriate position in the derivative instrument. Suppose the investor has a long position in asset 1, the result from Proposition 2 suggests that if the covariance is positive the risk averse investor will take a short position on the derivative, and if the covariance is negative he will take a long position.

To complete the characterization of the optimal financial portfolio, we now proceed to identify the different positions (long vs. short) that the investor will take on the first risky asset depending on the expected return. When an agent is allocating his wealth between a risk-free asset and one risky asset, a strictly positive risk premium is necessary and sufficient to obtain a strictly positive investment. In the next proposition we give a generalization of this result. In fact we can prove the following result.

**Proposition 3:** Let $\tilde{x}_1$ show either complete negative regression or complete positive regression on $\tilde{x}_2$. If $m_2 = 0$, then $\alpha_1^* \geq 0$ if and only if $E(\tilde{x}_1) \geq x_0$. In this case the position to take on $\tilde{x}_2$ (long vs. short) will depend on the covariance between $\tilde{x}_1$ and $\tilde{x}_2$.

**Proof of Proposition 3:**

Proving the first part of Proposition 3 is equivalent to proving that

$$\text{Sign}(\alpha_1^*) = \text{Sign}(m_1).$$
Since the agent is risk averse he will always prefer the certainty equivalent to a gamble with the same expected return. In fact with Jensen’s inequality, one has

\[ E(u(W(\alpha^*_t,0)))) \leq u(E(W(\alpha^*_t,0)))) = u(1 + x_0 + m_t \alpha^*_t). \]  

(6)

If \( \alpha^*_t \) and \( m_t \) have opposite signs then \( m_t \alpha^*_t < 0 \) and hence

\[ E(u(W(\alpha^*_t,0)))) < u(1 + x_0). \]  

(7)

The latter inequality contradicts the optimality of \((\alpha^*_t,0)\) since \((0,0)\) is a better investment strategy. Consequently, \( m_t \geq 0 \) is necessary and sufficient to obtain \( \alpha^*_t \geq 0 \). In addition, from Proposition 2, and since \( m_2 = 0 \), we know that \( \alpha^*_{2m} = 0 \). The optimal position to take on \( \tilde{x}_2 \) is then given by part b) of Proposition 1. \( Q.E.D. \)

Note that since a nil covariance is equivalent to independence in the class of regression dependence distributions (Lehmann (1966)), the position on \( \tilde{x}_1 \) will also depend on its risk premium even if the covariance between \( \tilde{x}_1 \) and \( \tilde{x}_2 \) is nil.
II. Examples

We now discuss additional examples of regression dependent distributions (for other examples see Lehmann (1959) and Tong (1980)).

Example 2. The second example is related to the set of distributions that allow for two-fund separation as defined by Ross (1978). We now show that this set is included in the broader set of regression dependent distributions. We know from Ross (1978) that under two-fund separation, \( \tilde{x}_1 \) and \( \tilde{x}_2 \) can be written as:

\[
\tilde{x}_i = x_0 + \beta_i (\tilde{x}_m - x_0) + \tilde{\varepsilon}_i, \text{ for } i = 1,2
\]  

(8)

where \( \tilde{x}_m \) is the return on the risky fund (or on any index) and \( \tilde{\varepsilon}_i \) is a random variable such that \( E(\tilde{\varepsilon}_i) = Cov(\tilde{\varepsilon}_i, \tilde{x}_m) = 0 \). Moreover, for two-fund separation to hold, we must verify the necessary and sufficient condition that \( E(\tilde{\varepsilon}_i / \tilde{x}_m) = 0, i = 1,2. \)

Under two-fund separation, the conditional distribution function can be written as:

\[
F(x_1 / x_2) = \Pr\left( \frac{\tilde{\varepsilon}_1 - \frac{\beta_1}{\beta_2} \tilde{\varepsilon}_2}{x_1 - x_0 - \frac{\beta_1}{\beta_2} (x_2 - x_0)} \leq 0 \right)
\]

(9)
As we can see, $F(x_1 / x_2)$ is always monotone in $x_2$ and the sign of this monotonicity depends on those of $\beta_1$ and $\beta_2$ which represent the sensitivity of each risky asset with respect to $\tilde{x}_m$. In fact, we have

$$\text{Sign} \left( \frac{\partial}{\partial x_2} F(x_1 / x_2) \right) = -\text{Sign} (\beta_1 \beta_2),$$

which, if we apply the result in Proposition 1, yields

$$\text{Sign}(\alpha_1^* \alpha_2^*) = \text{Sign}(\beta_1 \beta_2).$$

Example 2 shows that two-fund separating distributions generate regression dependence. An interesting question is the following: Is there any distribution that satisfies regression dependence but is not a two-fund separating distribution? The next example addresses this question.

**Example 3.** We now show that regression dependence does not imply two-fund separation. We consider the simple case of two risky assets with three states of the world: The corresponding returns are -3, 1 and 3 for the second risky asset and -2, 1 and 2 for the first risky asset. Table I gives the joint density of the returns.

(Table I about here)
Regression dependence can easily be proven (Lehmann, 1966). We consider two risk averse investors with preferences given respectively by 

\[ u_1(W) = W - \frac{1}{4} W^2 \]  
(with \( 1 - \frac{1}{2} W > 0 \) for all \( W \))

and 

\[ u_2(W) = \begin{cases} 
W - 1 & \text{if } W \leq 1 \\
0 & \text{if } W \geq 1
\end{cases} \]

and assume a zero risk-free interest rate. Note that investor \( u_2 \) is not in the Cass and Stiglitz (1970) family of separating functions. Otherwise, we would always have separation. The optimal investments for investor \( u_1 \) are \( (\frac{3}{5}, \frac{3}{8}) \). The optimal choice for investor \( u_2 \) is given by the semi-line \( 2\alpha_1^* + 3\alpha_2^* = 0 \) and \( \alpha_1^* \geq 0 \), which does not include the optimal choice of investor \( u_1 \) as illustrated in Figure 1. Two-fund separation is then not allowed by the distribution provided in Table I.

(Figure 1 about here)

**Example 4.** Suppose that \( (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_s) \) have a multinomial distribution corresponding to \( n \) trials and success probabilities \( (p_1, p_2, \ldots, p_s) \). The conditional distribution of \( \bar{x}_1 \) given \( \bar{x}_2 = x_2 \) is a binomial distribution and is negatively regression dependent (Lehmann, 1966).

**Other examples.** The Cauchy distribution (given that \( x_2 \in [0,1] \)) is a positive regression dependent distribution. The main difference between the normal distribution and the Cauchy distribution is the longer and flatter tails of the latter. An example of a negative regression dependent distribution is the bivariate Dirichlet. The Dirichlet extends the beta distribution to multivariate distributions. Finally, as shown by Tong (1980), regression dependence can be used for defining dependence by a mixture of distributions. The multivariate \( t \) is an example.
III. Conclusion

We have proposed the concept of regression dependence (Tukey (1958)) to analyze portfolio choice. This concept describes how two random variables behave together when they are simultaneously small or large. By assuming that the returns on risky assets are regression dependent, we were able to decompose the investment in one risky asset into a mean part based on the risk premium offered by the asset and a hedging part used against the fluctuations in the return on the other risky asset. This characterization of the optimal portfolio was done for all risk averse investors. Regression dependence was shown to be less restrictive than two-fund separating distributions (Ross (1978)). These results open the search for more general asset pricing models that allow for stochastic dependence going beyond the linear correlation as for the copula representation.

Several extensions of our article are possible. For example, we can consider a larger definition of dependence identified as quadrant dependence by Lehmann (1966) and verify whether the same type of decomposition between the mean and the hedging parts is possible. This generalization would have two advantages. First, quadrant dependence can be easily generalized to orthant dependence for dimensions higher than two. Moreover, Denuit and Scaillet (2001) provided two-test procedures for positive quadrant dependence: 1) specification of the dependence concepts in terms of distribution functions (see also Davidson and Duclos (2000)), and 2) exploitation of the copula representation. The two procedures did not reject the positive quadrant dependence of the data among US and Danish insurance claims. Mimouni (2002) applied the two-test procedures to
data on financial assets. As in Denuit and Scaillet (2001), he used a nonparametric approach to assess positive quadrant dependence. He did not reject positive quadrant dependence with the two-test procedures and consequently did not reject positive regression dependence. Further developments of these tests, for portfolios containing many stocks and derivatives, are open for future research.
Appendix

Proof of Proposition 1: The first-order condition (2) can be written as

\[ \frac{\partial}{\partial \alpha_2} E(u(W(\alpha_1, \alpha_2))) = \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} \left( x_2 - E(x_2) \right) u'(W(\alpha_1, \alpha_2)) \, dF(x_1 / x_2) \, dG(x_2) \]
\[ + m_2 \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} u'(W(\alpha_1, \alpha_2)) \, dF(x_1 / x_2) \, dG(x_2). \]  
(A1)

Let \( \alpha_{2h}^* \) be the solution to

\[ E\left( \tilde{x}_2 - E(\tilde{x}_2) \right) u'(W(\alpha_{1h}^*, \alpha_2)) = \text{Cov}(\tilde{x}_2, u'(W(\alpha_1, \alpha_2))) = 0 \]  
(A2)

then

\[ \frac{\partial}{\partial \alpha_2} E(\alpha_1, \alpha_{2h}^*) = m_2 \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} u'(W(\alpha_1, \alpha_{2h}^*)) \, dF(x_1 / x_2) \, dG(x_2). \]  
(A3)

It follows from the concavity of the objective function that \( \alpha_{2h}^* \geq \alpha_{2h} \) if and only if \( m_2 \geq 0 \), or that \( \alpha_{2h}^* - \alpha_{2h} \) has the same sign as \( m_2 \). Defining \( \alpha_{2m} = \alpha_2^* - \alpha_{2h}^* \) ends the proof of part a).

We now prove part b). The left-hand-side term in equation (A2) can be written after integration by parts as

\[ - \int_{\xi_1}^{\xi_2} \left[ \int_{\xi_1}^{\xi_2} (t - E(\tilde{x}_2)) dG(t) \right] I'(x_2) dx_2, \]  
(A4)

where \( I(x_2) = \int_{\xi_1}^{\xi_2} u'(W(\alpha_{1h}^*, \alpha_{2h}^*)) \, dF(x_1 / x_2) \) and \( I'(x_2) = \partial I / \partial x_2 \).

Since \( \int_{\xi_1}^{\xi_2} (t - E(\tilde{x}_2)) dG(t) \leq 0 \) for all \( x_2 \), it follows that if \( I(.) \) is monotonic then the sign of (A4) will be that of \( I'(.) \). The equality in (A2) that defines \( \alpha_{2h}^* \) will then be violated.

An integration by parts of the first derivative of \( I(.) \) gives
\[ I(x_2) = \alpha_{2h}^* \int_{\mathcal{D}_i} \left[ \frac{\partial}{\partial x_2} F(x_1/x_2) \right] dx_1 - \alpha_{2h}^* \int_{\mathcal{D}_i} u \left( W(\alpha_1^*, \alpha_{2h}^*) \right) \frac{\partial}{\partial x_2} F(x_1/x_2) dx_1 \]
\[ = \alpha_{2h}^* \left[ \frac{\partial}{\partial x_2} F(x_1/x_2) \right] - \alpha_{2h}^* \left[ u \left( W(\alpha_1^*, \alpha_{2h}^*) \right) \frac{\partial}{\partial x_2} F(x_1/x_2) \right] \]  

Consequently, a necessary condition for \( I(.) \) to be non monotonic is that

\[ \text{Sign}(\alpha_1^*, \alpha_{2h}^*) = \text{Sign} \left( \frac{\partial}{\partial x_2} F(x_1/x_2) \right). \]  

(A5)

In fact, suppose (A6) is not true. Then, by the assumption of regression dependence, we will necessarily have

\[ \text{Sign}(\alpha_1^*, \alpha_{2h}^*) = -\text{Sign} \left( \frac{\partial}{\partial x_2} F(x_1/x_2) \right). \]

Hence, \( \text{Sign}(I'(\cdot)) = \text{Sign}(\alpha_{2h}^* u) \), which will make \( I(.) \) a monotonic function. As discussed earlier this contradicts equality in (A2).

Finally, under our assumption of regression dependence, we have that

\[ \text{Sign}(\text{Cov}(\bar{x}_1, \bar{x}_2)) = -\text{Sign} \left( \frac{\partial}{\partial x_2} F(x_1/x_2) \right). \]  

(A7)

Equations (A6) and (A7) end the proof of part b) in Proposition 1. Q.E.D.
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This table presents the joint density function $f(x_1, x_2)$ of a regression dependent distribution. Regression dependence describes how two random variables behave together when they are simultaneously small or large. Here we observe that the two assets are positively regression dependent.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
<td>1/6</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
Figure 1. Optional solutions to Example 3. This figure depicts the optimal solution of investor $u_1$ at $(8/3, -5/3)$ and that of investor $u_2$ corresponding to the semi-line $2\alpha_1^* + 3\alpha_2^* = 0$ and $\alpha_1^* \geq 0$ when the data are from Table I. The joint distribution of this example does not yield a separating solution since the point $(8/3, -5/3)$ is not on the semi-line.