Measuring Prepayment Risk Exposure on Defaultable Callable Debt

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Preliminary. Comments welcome.

October 8, 2003

JEL Classification: G13, G21.

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Abstract

This paper provides a framework for measuring the lender’s exposure to prepayment risk on a defaultable callable debt contract within the Cox-Ingersoll-Ross term structure model. Prepayment risk stems from the lender’s inability to accurately anticipate the borrower’s call policy and impacts on the effectiveness of the lender’s interest rate risk management. Thus, the proposed measure of the magnitude of prepayment risk is the callable debt duration sensitivity to variations of the call threshold adopted by the borrower. We explicitly characterize the Macaulay duration and the stochastic duration sensitivities within the CIR term structure model combined with a reduced-form model of default risk. Simulations allow for assessing the relative effects of default risk, call moneyness, and interest rate risk on prepayment risk exposure. The paper derives testable implications to eventually determine empirically if there is a market premium for prepayment risk.
1 Introduction

Early repayment of the principal is an option commonly embedded in debt contracts. In efficient and frictionless capital markets, the option to repay the debt early should be regarded as a zero-sum provision as debtholders demand higher interest rates to compensate for the call option value. Moving away from the Modigliani-Miller paradigm however, the prepayment option may benefit borrowers allowing them to reduce their exposure to frictions. Thus, rationales for a call provision in debt contracts include enhanced managerial flexibility (Bodie and Taggart, 1978), tax advantages (Boyce and Kalotay, 1979), reduced agency conflicts (Barnea, Haugen and Senbet, 1980), and smaller exposure to interest rate risk (Kraus, 1983). Admittedly, if the frictions cited above are correctly identified and priced accordingly, the lender is no better- nor worse-off with a fairly-priced call provision. For instance, Dunn and Spatt (1986) and Mauer (1993) derive the borrower’s optimal refunding policy when there are third party costs. However, as first pointed out by Ingersoll (1977a), the exact pricing of the call provision requires that the borrower’s call policy is correctly anticipated. This may not be the case in practice, and the lender is then exposed to prepayment risk, defined as the uncertainty surrounding the borrower’s decision to exercise the early repayment option.

Prepayment risk has long been the focus of mortgage contract valuation (see e.g. the models by Stanton (1995) or Deng, Quigley and Van Order (2000)). Only recently has the scope of prepayment risk been extended

1In this case, the borrower’s optimal refunding policy (the one that maximizes the value of the call option) does not minimize the market value of the callable bond.

2Homeowners’ motivations for prepayment on mortgage debt include non-financial determinants, which makes the modelling of prepayment specific to these contracts.
to other types of debt, including corporate loans and bonds. For instance, Basel II’s new regulatory standards for interest rate risk management list embedded options (and in particular prepayment options) as one of the four sources of interest rate risk that have to be captured by a measurement system (along with repricing risk, yield curve risk, and basis risk). So far however, the importance of prepayment risk in corporate debt contracts has not been examined in depth. Indeed, most callable debt valuation models, such as Ingersoll (1977b), Artzner and Delbaen (1992), Kim, Ramaswamy and Sundaresan (1993) or Acharya and Carpenter (2002) to name a few, assume that the borrower makes the optimal call decision and do not address prepayment risk.\footnote{An exception is Ahn and Wilmott (1998) who analyze the exercise policy of an OTC American option when the holder maximizes a CARA, a HARA or a linear utility function. They find that the option writer gains from the difference between the price maximizing exercise time and the customer’s utility function maximizing exercise time. This analysis is not applied to corporate debt contracts however.} However, it is well documented that callable debt is often called suboptimally. Vu (1986) and King and Mauer (2000) provide strong evidence of delays before and after the optimal call date for non-convertible corporate bonds.

Some rationales for early or late calls have been explored. Smith and Warner (1979) argue that debt may be called early to get rid of restrictive covenants on the firm’s investment policy. Kraus (1983), Fisher, Heinkel and Zechner (1989) and Mauer (1993) argue that calls are delayed because of refunding costs. However, as pointed out by Longstaff and Tuckman (1994), the refunding motive may not be the only factor at stake. They suggest that optimal call policy is delayed in the presence of multiple lenders. Indeed, if a bond is called without being refinanced, the other bondholders will benefit from the reduction in the firm’s total default risk. Thus, the firm will delay
the call to trade-off this wealth transfer to remaining bondholders. In sum, the optimal call policy may be biased for at least three reasons: the possibility to seize investment opportunities, the costs associated with refunding, and the wealth transfers to other claimholders. In this paper, we shall assume that the lender cannot accurately assess these three factors, and is therefore exposed to prepayment risk. To justify this, we may think of an investment opportunity set that is partly unknown to the lender. In addition, the magnitude of refunding costs as well as that of wealth transfer effects depend on the new leverage ratio that the borrower will adopt after calling the original debt. Again, we may think that this financing decision is partly unknown to the lender.

Prepayment risk affects the lender’s management of interest rate risk. In particular, banks often target specific durations on debt instruments in order to match the interest rate risk exposure of their assets and liabilities. This matching will be jeopardized if prepayment risk alters the duration on some of the lender’s callable assets. This suggests that duration variations due to unexpected changes in the borrower’s call policy may serve as a relevant measure of prepayment risk magnitude. In this spirit, the aim of this paper is to provide a framework in which prepayment risk magnitude can be easily measured.\footnote{Another line of research, followed by Dunn and Spatt (1999), assumes that no information on the borrower’s prepayment likelihood is available, and derives general restrictions on callable debt contracts that are robust across all possible pricing frameworks.} To achieve this, we depart from the standard contingent claims model of the levered firm. In such a framework, the borrower’s asset value is common knowledge and, consequently, so are his investment and financing decisions. Thus, there is no reason why the lender could not assess the factors that would bias the borrower’s call policy. In other words, there
could be a theoretical inconsistency in measuring prepayment risk within a structural model using asset value as the observable state variable. By contrast, we shall rely on an intensity-based approach of default risk, initiated by Jarrow and Turnbull (1995), along with the Cox-Ingersoll-Ross term structure model. Within this framework, Barone, Barone-Adesi and Castagna (1998) obtain pricing formulae for defaultable bonds, European options on defaultable bonds as well as vulnerable European options. To address and measure prepayment risk on defaultable callable debt, we rely on results concerning first hitting times of square-root processes that were recently derived by Göing-Jaeschke and Yor (1999). Specifically, we derive quasi-analytical expressions for the Macaulay duration and for the stochastic duration on defaultable callable debt. We then determine numerically how these durations are affected by variations of the borrower’s call threshold around the traditional “optimal” boundary, i.e. the one that maximizes the value of the call option in the absence of market frictions. This duration sensitivity is our proposed measure for prepayment risk magnitude. Although the Macaulay duration has serious limitations, it is still widely used in the industry, so we shall report prepayment risk measures for both the Macaulay and the stochastic durations.

The paper is organized as follows. In section 2, we develop the valuation framework. Section 3 presents simulation-based results on the effects of interest rate risk, call moneyness, and default risk on prepayment risk magnitude. Section 4 concludes. Technical developments and figures are gathered in the appendix.
2 The valuation framework

2.1 Assumptions

We consider an economy where assets are continuously traded in arbitrage-free and complete markets. We know from Harrison and Pliska (1981) that asset prices discounted at the risk-free rate are martingales under the unique risk-neutral probability measure $\mathbb{Q}$. Following Jarrow and Turnbull (1995) or Barone, Barone-Adesi and Castagna (1998), the borrower’s timing of default follows an exponential distribution with a constant hazard rate (intensity of default). Under the risk-neutral measure, the hazard rate is denoted by $\eta$. Thus, the probability function of $\theta$, the random date of the borrower’s default, is given by

$$
\mathbb{Q}(\theta > t) = e^{-\eta t}.
$$

In regards to this specification, default risk is not a decision variable, which implies moral hazard problems are ruled out. It is well known from Jarrow and Turnbull (1995) or Duffie and Singleton (1999) that the borrower’s credit spread is the risk-neutral default intensity $\eta$. In other words, the borrower’s refinancing market rate, $R_t$, is the default-risk-adjusted interest rate given by

$$
R_t = r_t + \eta
$$

where $r_t$ stands for the money market rate.\footnote{Duffie and Singleton (1999) and Madan and Unal (2000) provide extensions of this framework allowing for a stochastic hazard rate.}

\footnote{A simple proof goes as follows. The present value of one risky dollar promised at $\Delta t$ units of time from now is given by}

$$
\frac{1}{1 + R \Delta t} = \frac{(1 - \eta \Delta t) \cdot 1 + \eta \Delta t \cdot 0}{1 + r \Delta t} \iff R \Delta t = (r + \eta) \Delta t + R \eta (\Delta t)^2.
$$
Interest rate risk is captured with the Cox-Ingersoll-Ross (thereafter CIR) term structure model. Specifically, we assume that \( \{r_t\}_{t \geq 0} \) follows a square-root mean-reverting process under \( Q \), i.e.

\[
dr_t = \alpha (\beta - r_t) \, dt + \sigma \sqrt{r_t} \, dW_t
\]

where \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion. The instantaneous risk-free rate reverts to the long-run mean \( \beta \) with the speed reversion \( \alpha \). We require that \( \sigma^2 \leq 2\alpha\beta \) for the process \( \{r_t\}_{t \geq 0} \) to remain strictly positive.

### 2.2 Debt values

In this setting, we now derive the values of the four defaultable debt securities: callable and non-callable, discount and coupon-bearing. Let \( B(t, r) \) denote the value of the defaultable non-callable discount bond with nominal 1, maturity \( t \), and interest rate current value \( r \). Similarly, \( B(c, t, r) \) denotes the value of the defaultable non-callable debt paying the continuous coupon \( c \). From a direct extension of Cox, Ingersoll and Ross (1985), it is well known that (see e.g. Barone et al. (1998))

\[
B(t, r) = a(t) \exp(-\eta t - b(t) r)
\]

with

\[
a(t) = \left( \frac{2\gamma e^{0.5(\gamma + \alpha + \mu)t}}{(\gamma + \alpha + \mu)(e^{\gamma t} - 1) + 2\gamma} \right)^{\frac{2\alpha\beta}{\sigma^2}}
\]

\[
b(t) = \frac{2}{\alpha + \mu + \gamma \coth\left(\frac{2\gamma t}{\sigma^2}\right)}
\]

\[
\gamma = \sqrt{(\alpha + \mu)^2 + 2\sigma^2}
\]

By taking the limit for \( \Delta t \to 0 \), we obtain the continuously compounded risk-adjusted short rate process \( R = r + \eta \).
and $\mu$ stands for the interest rate risk premium. Moreover, since default parameters are exogenous, $B(c, t, r)$ may be expressed as a linear combination of defaultable discount bonds.\footnote{As noticed by Duffie and Singleton (1999), the linear property does not hold in the general case where default parameters depend on interest rates.}

We now turn to callable debt securities. For tractability, we will focus on perpetuities.\footnote{The extension to finite time horizon is a non-trivial exercise since the optimal call policy can no longer be derived in an analytical way.} Specifically, we consider a defaultable callable debt consol with continuously paid coupon $c$. The value of this contract is denoted by $CB(c, \infty, r)$. In case of prepayment, the holder receives the strike price of the call option $K$, known at contract inception. We denote by $\tau$ the random date when the debt is called. Within an infinite time horizon, $\tau$ may be written as

$$\tau = \inf \{ u \geq 0 : r_u = h \}.$$ 

In our setting, the level $h$ is chosen by the borrower and the lender cannot determine this level with full accuracy. What he can determine however is the “optimal” call threshold $h^*$. This is the level that maximizes the value of the call option in the absence of market frictions. The following proposition gives the value of the defaultable callable debt under the optimal call policy (see Appendix A for a proof).

**Proposition 1** Assuming the borrower will maximize the value of the call option, the value the defaultable callable coupon-bearing perpetuity is given by

$$CB(c, \infty, r) = B(c, \infty, r) - [B(c, \infty, h^*) - K] \exp[\phi(r - h^*)]$$

where

$$h^* = \arg \max_h \left[ (B(c, \infty, h) - K) e^{\phi(r-h)} \right]$$

(1)
\[ \phi = \frac{1}{r \sigma^2} \left( (\alpha + \mu) r - \alpha \beta - \sqrt{[\alpha \beta - (\alpha + \mu) r]^2 + 2\sigma^2 r (r + \eta)} \right). \]

2.3 Yield and durations

In this setting, we now derive quasi-analytical expressions for durations of defaultable callable debt, that will serve for our measure of prepayment risk exposure. First, the expected yield on the defaultable callable debt and the corresponding Macaulay duration are given by the following proposition (see Appendix B for a proof).

**Proposition 2** The expected yield on the defaultable callable coupon-bearing perpetuity is the solution to

\[ CB(c, \infty, r) = \frac{c}{y + \eta} + \left( K - \frac{c}{y + \eta} \right) \mathcal{L}_{y + \eta} (\tau). \]  

(2)

where \( CB(c, \infty, r) \) is given by equation (1), \( \mathcal{L}_x(\tau) \) stands for the Laplace transform of the random call date \( \tau \) with rate \( x \), i.e.

\[ \mathcal{L}_x(\tau) = \int_0^\infty e^{-xt} f(t) \, dt \]

and \( f(.) \) is the density function of \( \tau \). The Macaulay duration of this instrument is given by

\[ Dur_m = \frac{1}{CB(c, \infty, r)} \left[ \frac{c (1 - \mathcal{L}_{y+\eta}(\tau))}{(y^* + \eta)^2} + \left( \frac{c}{y^* + \eta} - K \right) \left. \frac{\partial \mathcal{L}_{y+\eta}(\tau)}{\partial y} \right|_{y=y^*} \right] \]

(3)

where \( y^* \) is the solution to equation (2).

Equation (2) states that the value of defaultable callable debt equals the value of a perpetuity discounted at the risk-adjusted yield \( y + \eta \) plus the
incremental value upon call \( (K - \frac{c}{y+\eta}) \) times the Arrow-Debreu price of call \( (L_{y+\eta}(\tau)) \). Proposition 2 gives analytical solutions for the yield and the duration of the defaultable callable debt, provided the Laplace transform of the call date is known. Within the CIR framework, Göing-Jaeschke and Yor (1999) show that the Laplace transform of the first hitting time of \( r_t \) starting at \( x \) to the fixed boundary \( h \), \( h < x \), with rate \( z \) is given by\(^9\)

\[
\mathcal{L}_z(\tau) = \frac{\psi \left( \frac{z^2}{4\alpha}, \frac{2(\sigma^2-\alpha\beta)}{\sigma^2}, \frac{2\alpha x}{\sigma^2} \right)}{\psi \left( \frac{2\sigma^2 - 2\alpha\beta}{\sigma^2}, \frac{2\alpha x}{\sigma^2} \right)}
\]

with \( \psi(a, b; w) \) defined by

\[
\psi(a, b; w) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \phi(a, b; w) + \frac{\Gamma(b-1)}{\Gamma(a)} w^{1-b} \phi(1+a-b, 2-b; w)
\]

where \( \phi(a, b; w) \) is the confluent hypergeometric function\(^{10}\) and \( \Gamma(x) \) is the gamma function.

Next, we characterize the stochastic duration, by extending the result of Cox, Ingersoll and Ross (1979) for defaultable callable debt (see Appendix C for a proof).

**Proposition 3** The stochastic duration of the defaultable callable coupon-bearing perpetuity is given by

\[
Dur_s = b^{-1} \left( c \int_0^\infty b(t) B(t, r) dt + (\phi + (r-h) \phi_r) [B(c, \infty, h) - K] e^{\phi(r-h)} \right).
\]

\(^{9}\)It should be noted that very few results concerning first hitting times of stochastic processes that are commonly used in term structure models are available. For example, the law of the first hitting time of the CIR process has no known explicit solution. As for the Vasicek process, Göing-Jaeschke and Yor (2003) show the law of its first hitting time to a fixed boundary derived by Leblanc et al. (2000) is only valid for the boundary 0.

\(^{10}\)The confluent hypergeometric function also appears in the models by Merton (1974) and Black and Cox (1976) to price coupon-bearing risky debt.
where $\phi$ is defined in proposition 1 and $\phi_r = \partial \phi / \partial r$.

2.4 The magnitude of prepayment risk

We now measure the prepayment risk magnitude, denoted by $PRM_i$, as

the sensitivity of the Macaulay duration ($i = m$) or that of the stochastic
duration ($i = s$) to deviations of the borrower’s call policy around the optimal
call policy. Formally, this measure may be defined as

$$PRM_i = -\frac{d(Dur_i)}{dh} \bigg|_{h=h^*}, i = m, s$$

and, in the absence of an analytical expression for this derivative, it will be
approximated by

$$PRM_i \simeq -\lim_{\varepsilon \to 0} \frac{Dur_i(h^* + \varepsilon) - Dur_i(h^* - \varepsilon)}{2\varepsilon}, i = m, s.$$ 

The negative sign is to ensure the positivity of our measure.

Although $PRM_i$ is defined as a local measure of duration variations
around the optimal call threshold, it also appears as a reasonably good ap-
proximation of how duration changes in case the borrower strongly departs
from the optimal call policy. Indeed, the duration is almost linear in the call
threshold $h$, as will be shown below.

3 Simulations analysis

3.1 Duration measures and prepayment risk

Before investigating the prepayment risk exposure, we first emphasize the
impact of the choice of duration on interest rate risk measurement, and,  
ultimately, on prepayment risk measurement. As already pointed out by
Cox, Ingersoll and Ross (1979), Macaulay duration can substantially differ from stochastic duration, especially for long-lived securities. We show that a similar result holds for defaultable callable debt. We consider a base case characterized by: $c = 10, K = 110, r = 0.07, \alpha = 0.1, \beta = 0.08, \text{ and } \sigma = 0.06$. Parameters $r$ and $\beta$ correspond to the current and the equilibrium level of the riskless interest rate. Speed reversion coefficient $\alpha$ and volatility $\sigma$ complete the interest rate dynamics. Their parametrization is in line with estimates commonly obtained by empirical tests of the CIR (1985) model.\footnote{See for instance Chan et alii (1992) and de Jong and Santa-Clara (1999). Without any loss of generality, we assume that interest rate risk premium $\mu = 0$.}

Setting $\eta = 0.02$, the defaultable non-callable perpetuity is worth $B(c, \infty, r) = 109.89$ so that the call option is very close to the money.

Table 1 reports Macaulay and stochastic durations for three debt issues that differ in the moneyness of their embedded call option.\footnote{Tables and figures are reported in the appendix D.} Durations are calculated with increasing levels of default risk, while controlling for the degree of moneyness. Early termination of the debt contract gets more likely as default risk or call moneyness increase, and the two duration measures decrease accordingly. However, stochastic duration is in all cases above Macaulay duration. From table 1, we see the relative difference is all the higher since the call is deep in the money. Because Macaulay duration captures the impact of a parallel shift in the entire yield curve, it overestimates the impact of interest rate risk on the probability of call exercise.

As can be expected, different interest rate risk measures yield different prepayment risk measures. With the base case parameters, the optimal call policy is to call as soon as $r = h^* = 3.363\%$, the Macaulay duration is
3.948, and $PRM_m = 151.26$. This means that, if this borrower opts to call later than the optimal rule, by calling say 10 basis points below $h^*$, then the Macaulay duration on defaultable callable debt will increase by around $151.26 \times 0.001 \simeq 0.151$, that is a bit less than two months. As for stochastic duration, we have that $Dur_s = 6.223$ and $PRM_s = 77.43$. The same deviation from the optimal call policy will cause the stochastic duration to decrease by a bit less than one month. As a matter of fact, using Macaulay duration as the measure of interest rate risk leads to an overestimation of prepayment risk exposure.

As we mentioned earlier, $PRM_i$ stands as a reliable measure of prepayment risk magnitude also for large deviations around the optimal call policy. With the base case again, suppose the borrower calls 100 basis points below $h^*$. According to $PRM_m$ and $PRM_s$, the Macaulay and stochastic durations should increase by 1.51 (more than a year and a half) and 0.77 (more than nine months) respectively. In fact, we have that $Dur_m(2.363\%) = 5.539$ and $Dur_s(2.363\%) = 7.034$ which means the exact increases of duration are underestimated by less than 5% in both cases.

### 3.2 Interest rate environment and prepayment risk

We turn to the analysis of prepayment risk magnitude. Within our base case, $PRM_i$ is calculated for various interest rate dynamics. Results are presented in Figures 1 (Macaulay duration) and 2 (stochastic duration). All these figures also report the corresponding duration and optimal call threshold.

For both durations, $PRM_i$ decreases with the current level of interest rate. If $r$ is high indeed, the call option is deep out-of-the-money. If the borrower is to deviate from the optimal call policy, this will not happen in the near future so that the impact of the lender’s cash inflows is small. As a
result, the exposure to prepayment risk will be limited. Note also that both durations increase with \( r \), indicating that the call effect more than offsets the discounting effect on duration. Both durations also agree on the effect of speed reversion. In our base case, the interest rate is currently at 7% and reverts to the level of 8%. A higher speed reversion therefore implies less uncertainty around this trend. This results in a lower call exercise probability. Macaulay and stochastic durations increase with \( \alpha \), thereby capturing this effect. Again, since duration of non-callable debt is decreasing with \( \alpha \) (see e.g. Cox, Ingersoll and Ross (1979)), simulations indicate that the call effect dominates the discounting effect.

Interestingly, prepayment risk exposure is little affected by interest rate volatility. An increase in \( \sigma \) reflects a higher interest rate risk and a higher call exercise probability. This has two opposite effects on duration. Macaulay duration is increasing and concave in \( \sigma \), underscoring the call moneyness effect. By contrast, stochastic duration assigns equal weights to these two effects and is very little affected by changes in volatility. Both durations lead to a measure of prepayment risk exposure that is also almost invariant in \( \sigma \). Note however that since Macaulay duration only captures parallel shifts of the whole yield curve, it only considers extreme interest rate movements that have a dramatic impact on duration sensitivity to changes in the call policy. As a result, \( PRM_m \) stands twice as high as \( PRM_s \).

The sensitivity with respect to the last interest rate parameter, the long run mean \( \beta \), confirms again that Macaulay and stochastic duration do not capture the effect of callability in the same manner. Macaulay duration is strictly increasing with \( \beta \) and therefore reflects a dominating call effect on the discounting effect. By contrast, stochastic duration is a humped function of \( \beta \), indicating that the call effect tends to fade away as the option gets deep
out-of-the-money. As a result, once stochastic duration is no longer affected by callability for high a β, it becomes less and less sensitive to deviations from the optimal call policy. This explains why $PRM_s$ converges to zero, whereas $PRM_m$ keeps increasing with β since the call option still impacts on Macaulay duration.

### 3.3 Prepayment risk, call likelihood and default risk

In figure 3, we simulate durations and prepayment risk magnitudes as a function of default risk intensity η. This figure completes the analysis from table 1. When controlling for call moneyness, both durations decrease with default risk. However, a pure increase in η also makes the call option less in-the-money, which raises duration. Figure 3 indicates the combination of these two effects. According to both duration measures, the call effect more than offsets the default risk effect, since $Dur_m$ and $Dur_s$ increase with η.

However, the trade-off is not the same for prepayment risk magnitudes. Stochastic duration is increasing and slightly concave in η, indicating that the call effect tends to fade away as default risk rises. Consequently, very risky debt issues exhibit less exposure to prepayment risk. The opposite result obtains if one uses Macaulay duration.

### 3.4 Prepayment risk and debt design

Finally, we investigate how the design of the callable debt contract can affect its exposure to prepayment risk. In our setting, the debt contract is fully characterized by the coupon rate and the call strike. Figure 4 reports durations and prepayment risk magnitudes for various levels of $K$. All other things equal, the call moneyness decreases with the strike, which lengthens
the expected lifetime of the debt issue. As a matter of fact, Macaulay and stochastic durations both increase with $K$. Irrespective of the call probability, a higher $K$ also means a higher cash inflow for the lender, so duration should be more affected by a change in the call policy. Figure 4 shows that Macaulay duration puts more emphasis on this second effect as $PRM_m$ increases with $K$. According to stochastic duration however, even if the strike $K$ may represent a high cash inflow, its impact on duration sensitivity is completely offset by the decreasing call probability. Consequently, $PRM_s$ decreases with $K$ or, equivalently, debt issues with out-of-the-money call options are less exposed to prepayment risk.

Figure 5 reports durations and prepayment risk magnitudes for various levels of coupon payment. Duration of non-callable debt issue decreases with the coupon rate. This effect is reinforced with callability since, all other things equal, the call option gets more in-the-money as $c$ increases. Again, stochastic duration puts more emphasis on the call effect, as it recognizes that for a high coupon rate, duration will be significantly affected by the in-the-money call and will therefore be highly sensitive to any changes in the borrower’s call policy.

4 Conclusion and extensions

We have presented a simple framework that allows for measuring the lender’s exposure to prepayment risk on a defaultable callable debt contract. In our setting, this exposure is a function of combined interest rate risk, default risk, and call moneyness. Overall, results suggest that stochastic duration captures more accurately the optionality of callable debt than Macaulay duration does, and consequently provides a more reliable basis for measuring
prepayment risk exposure. According to this duration measure, defaultable callable debt issues will be more exposed to prepayment risk if current interest rate, interest rate equilibrium value, default risk intensity, and strike price are low, or if interest rate speed reversion and coupon rate are high. Also, prepayment risk magnitude is little affected by interest rate volatility.

Interestingly, some determinants of prepayment risk exposure are market factors but some others (in particular those pertaining to debt design) are manageable by the lender. This suggests that banks may implement an active strategy of prepayment risk exposure reduction, in line with Basel II’s new regulatory standards.

Perhaps one of the most interesting question arising from our work is: Is there a market price for prepayment risk? Our paper provides testable implications to address this issue. Using the properties we have identified, and controlling for other factors that affect debt pricing (such as liquidity or tax issues), it is possible to investigate whether or not defaultable callable debt contracts with a higher exposure to prepayment risk sell at a significant premium.
References


Appendix A

Proof of proposition 1

The optimal call policy consists in calling the debt so as to maximize the perpetual American call value, denoted by $g(\infty, r)$, written on the borrower’s financing instrument with strike $K$. This call option entitles the borrower the right to put an end to her current defaultable callable consol with coupon $c$ by paying the strike. The payoff upon exercise of the option to call the debt is therefore given by

$$
c\mathbb{E} \left[ \int_0^\infty \exp \left( -\int_0^t r_s ds \right) 1_{t<\theta} dt \bigg| r = h \right] - K = c \int_0^\infty B(t, h) dt - K
$$

where $h$ stands for the call threshold and $B(t, h)$ is the price of the defaultable zero-coupon bond with nominal 1, maturity $t$ with initial interest rate value $h$.

Using Itô’s lemma and standard arbitrage arguments, $g$ satisfies the following ODE (see e.g. Barone et al. (1998))

$$
[\alpha \beta - (\alpha + \mu) r] g_r + \frac{1}{2} \sigma^2 r g_{rr} = (r + \eta) g
$$

where $g_r$ denotes the partial derivative of $g$ with respect to $r$. The general solution is

$$
g(\infty, r) = ae^{\psi r} + be^{\phi r}
$$

with $a$ and $b$ two constants and

$$
\begin{align*}
\psi &= \frac{1}{r\sigma^2} \left( (\alpha + \mu) r - \alpha \beta + \sqrt{[\alpha \beta - (\alpha + \mu) r]^2 + 2\sigma^2 r (r + \eta)} \right) > 0 \\
\phi &= \frac{1}{r\sigma^2} \left( (\alpha + \mu) r - \alpha \beta - \sqrt{[\alpha \beta - (\alpha + \mu) r]^2 + 2\sigma^2 r (r + \eta)} \right) < 0
\end{align*}
$$

Boundary condition $g(\infty, \infty) = 0$ implies that $a = 0$. The second boundary condition is the payoff upon exercise

$$
g(\infty, h) = B(c, \infty, h) - K = be^{\phi h}
$$
and this yields

\[ b = [B(c, \infty, h) - K] e^{-\phi h}. \]

Hence, we obtain

\[ g(\infty, r) = [B(c, \infty, h) - K] e^{\phi(r-h)}. \]

The optimal call threshold \( h^* \) maximizes the American call value across all possible first hitting times. It therefore verifies

\[ h^* = \arg \max_{h} (B(c, \infty, h) - K) e^{\phi(r-h)}. \]

Finally, the value of the defaultable callable coupon-bearing perpetuity is that of the equivalent non-callable perpetuity less the value of the call option

\[ CB(c, \infty, r) = B(c, \infty, r) - g(\infty, r), \]

which completes the proof.
Appendix B

Proof of proposition 2

The callable debt yield is solution to

$$CB(c, \infty, r) = cE \left( \int_{0}^{\inf(\tau, \theta)} e^{-yt} dt \right) + K E \left( e^{-y\tau 1_{\theta > \tau}} \right),$$

where $1_{\omega}$ denotes the indicator function of the event $\omega$. Solving for the integrals and rearranging terms yields

$$CB(c, \infty, r) = \frac{c}{y} - \frac{c}{y} E \left( e^{-y\tau 1_{\theta > \tau}} \right) - \frac{c}{y} E \left( e^{-y\theta 1_{\tau > \theta}} \right) + K E \left( e^{-y\tau 1_{\theta > \tau}} \right). \quad (*)$$

We immediately get

$$E \left( e^{-y\tau 1_{\theta > \tau}} \right) = L_{y+\eta}(\tau).$$

Introducing the density of $\theta$ (an exponential law with parameter $\eta$), we may write the second expectation of equation $(*)$ as

$$E \left( e^{-y\theta 1_{\tau > \theta}} \right) = E \left( \int_{0}^{\tau} e^{-yu} e^{-\eta u} du \right) = \frac{\eta}{y + \eta} [1 - L_{y+\eta}(\tau)].$$

Hence, equation $(*)$ becomes

$$CB(c, \infty, r) = \frac{c}{y} + \left( K - \frac{c}{y} \right) L_{y+\eta}(\tau) - \frac{c}{y} \frac{\eta}{y + \eta} [1 - L_{y+\eta}(\tau)]$$

$$= \frac{c}{y + \eta} + \left( K - \frac{c}{y + \eta} \right) L_{y+\eta}(\tau).$$

The Macaulay duration $Dur_m$ is defined by

$$Dur_m = -\frac{dCB(c, \infty, r)}{dy} \frac{1}{CB(c, \infty, r)}.$$

Using the expression of $CB(c, \infty, r)$ in equation (2) completes the proof.
Appendix C

Proof of proposition 3

Following Cox, Ingersoll and Ross (1979), the stochastic duration of a bond is defined as the maturity of a discount bond with the same risk. In the case of defaultable callable debt, this yields

\[ Dur_s = F^{-1} \left( -\frac{CB_r(c, \infty, r)}{CB(c, \infty, r)} \right), \]

where \( F = -B_r(t, r) / B(t, r) \).\(^{13}\) In the CIR term structure model and in the Barone et alii (1998) framework, the reciprocal function of \( F \) is explicit:

\[ F^{-1}(x) = b^{-1}(x) = \frac{2}{\gamma} \coth^{-1} \left( \frac{2}{\gamma x} - \frac{\alpha + \mu}{\gamma} \right). \]

From appendix A, we have that

\[ Dur_s = b^{-1} \left( -\frac{B_r(c, \infty, r) - g_r(\infty, r)}{CB(c, \infty, r)} \right), \]

which yields

\[ Dur_s = b^{-1} \left( c \int_0^\infty b(t) B(t, r) dt + (\phi + (r - h) \phi_r) [B(c, \infty, h) - K] e^{\phi(r-h)} \right) / CB(c, \infty, r). \]

\(^{13}\)Strictly speaking, we have \( F = -CB_r(t, r) / CB(t, r) \). However, \( CB(t, r) = B(t, r) - g(t, r) \) with \( g(t, r) \) the value of the American call on \( B(t, r) \) with same maturity \( t \) and strike \( K \). Since the underlying has no interim payments, the American call has no early exercise premium, and we get \( g(t, r) = (1 - K) B(t, r) \). Consequently, we have that \( F = -CB_r(t, r) / CB(t, r) = -B_r(t, r) / B(t, r) \).
Appendix D

Tables

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Table 1: Duration measures, default risk and call moneyness.

Base case interest rate parameters are: $r = 0.07$, $\alpha = 0.1$, $\beta = 0.08$, and $\sigma = 0.06$. Coupon payment is $c = 10$. For issue 1, the call is out-of-the money, i.e. $K$ is set such that $B(c, \infty, r) \cdot 1.1 = K$. For issue 2, the call is at-the money, i.e. $K$ is set such that $B(c, \infty, r) = K$. For issue 3, the call is in-the money, i.e. $K$ is set such that $B(c, \infty, r) \cdot 0.9 = K$. Macaulay and stochastic durations are reported in years for various levels of default risk.
Figure 1: Call threshold, Macaulay duration and prepayment risk magnitude for various interest rate environments.

Base case interest rate parameters are: \( r = 0.07, \alpha = 0.1, \beta = 0.08, \) and \( \sigma = 0.06. \) Debt contract parameters are: \( c = 10, K = 110. \) Default risk is: \( \eta = 0.02. \) On the horizontal axis is respectively the interest rate speed reversion (\( \alpha \)), long term mean (\( \beta \)), volatility (\( \sigma \)), and current value (\( r \)). The optimal call threshold (\( h^* \)) is expressed in percentage on the left scale and is plotted with a dashed line. The Macaulay duration (\( Dur_m \)) is expressed in years on the left scale and is plotted with a thin line. The prepayment risk magnitude (\( PRM_m \)) is expressed in units on the right scale.
Figure 2: Call threshold, stochastic duration and prepayment risk magnitude for various interest rate environments.

Base case interest rate parameters are: \( r = 0.07 \), \( \alpha = 0.1 \), \( \beta = 0.08 \), and \( \sigma = 0.06 \). Debt contract parameters are: \( c = 10 \), \( K = 110 \). Default risk is: \( \eta = 0.02 \). On the horizontal axis is respectively the interest rate speed reversion (\( \alpha \)), long term mean (\( \beta \)), volatility (\( \sigma \)), and current value (\( r \)). The optimal call threshold (\( h^* \)) is expressed in percentage on the left scale and is plotted with a dashed line. The stochastic duration (\( Dur_s \)) is expressed in years on the left scale and is plotted with a thin line. The prepayment risk magnitude (\( PRM_s \)) is expressed in units on the right scale.
Figure 3: Call threshold, Macaulay and stochastic durations and prepayment risk magnitudes for various levels of default risk.

Base case interest rate parameters are: $r = 0.07$, $\alpha = 0.1$, $\beta = 0.08$, and $\sigma = 0.06$. Debt contract parameters are: $c = 10$, and $K = 110$. On the horizontal axis is the hazard rate of default risk ($\eta$). The optimal call threshold ($h^*$) is expressed in percentage on the left scale and is plotted with a dashed line. The durations ($Dur_{i}, i = m, s$) are expressed in years on the left scale and are plotted with a thin line. The prepayment risk magnitudes ($PRM_{i}, i = m, s$) are expressed in units on the right scale and are plotted with a thick line.
Figure 4: Call threshold, Macaulay and stochastic durations and prepayment risk magnitudes for various levels of call strike.

Base case interest rate parameters are: $r = 0.07, \alpha = 0.1, \beta = 0.08,$ and $\sigma = 0.06$. Debt contract parameters are: $c = 10$. Default risk is: $\eta = 0.02$. On the horizontal axis is the call strike ($K$). The optimal call threshold ($h^*$) is expressed in percentage on the left scale and is plotted with a dashed line. The durations ($Dur_i, i = m, s$) are expressed in years on the left scale and are plotted with a thin line. The prepayment risk magnitudes ($PRM_i, i = m, s$) are expressed in units on the right scale and are plotted with a thick line.
Figure 5: Call threshold, Macaulay and stochastic durations and prepayment risk magnitude for various levels of coupon.

Base case interest rate parameters are: $r = 0.07$, $\alpha = 0.1$, $\beta = 0.08$, and $\sigma = 0.06$. Debt contract parameters are: $K = 110$. Default risk is: $\eta = 0.02$. On the horizontal axis is the continuous coupon payment $(c)$. The optimal call threshold $(h^*)$ is expressed in percentage on the left scale and is plotted with a dashed line. The durations $(Dur_i, i = m, s)$ are expressed in years on the left scale and are plotted with a thin line. The prepayment risk magnitudes $(PRM_i, i = m, s)$ are expressed in units on the right scale and are plotted with a thick line.