Near-Rational Exuberance

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Abstract

We study how the use of judgement or “add-factors” in macroeconomic forecasting may disturb the set of equilibrium outcomes when agents learn using recursive methods. We isolate conditions under which new phenomena, which we call exuberance equilibria, can exist in standard macroeconomic environments. Examples include a simple asset pricing model and the New Keynesian monetary policy framework. Inclusion of judgement in forecasts can lead to self-fulfilling fluctuations, but without the requirement that the underlying rational expectations equilibrium is locally indeterminate. We suggest ways in which policymakers might avoid unintended outcomes by adjusting policy to minimize the risk of exuberance equilibria. JEL codes: E520, E610.

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1 Introduction

1.1 Judgement variables in forecasting

Judgement is a fact of life in macroeconomic forecasting. It is widely understood that even the most sophisticated econometric forecasts are adjusted before presentation. This adjustment is so pervasive that it is known as the use of “add-factors”—subjective changes to the forecast which depend on the forecaster’s assessment of special circumstances that are not well summarized by the variables that are included in the econometric model. A forthright discussion of how prominently judgement enters into actual macroeconomic forecasting is contained in Reifschneider, Stockdon, and Wilcox (1997). As they state, “... [econometric] models are rarely, if ever, used at the Federal Reserve without at least the potential for intervention based on judgement. Instead, [the approach at the Federal Reserve] involves a mix of strictly algorithmic methods (“science”) and judgement guided by information not available to the model (“art”) (p. 2, italics in original). Recently, some authors have argued that economic theory needs to take explicit account of the effects of judgement on the behavior of macroeconomic systems.¹

We wish to think of the news or add-factor that modifies the forecast as a qualitative, unique, commonly understood economy-wide variable: In sum, a judgement variable. An example of a judgemental adjustment is suggested by Reifschneider, et. al. (1997), when they discuss the “financial headwinds” that were thought to be inhibiting U.S. economic growth in the early to mid-1990s. As they discuss, the headwinds add-factor was used to adjust forecasts over a period of many quarters. It was communicated to the public prominently in speeches by Federal Reserve Chairman Alan Greenspan. It was thus widely understood throughout the economy and was highly serially correlated. This is the type of variable we have in mind, although by no means would we wish to restrict attention to this particular example. We

think add-factoring is occurring continuously.

We take it for granted that it is a matter of conventional wisdom among economists that judgement is all to the good in macroeconomic forecasting. Models are, of course, crude approximations of reality and must be supplemented with other information not contained in the model.

1.2 Feedback from judgement

Our focus in this paper is on how the add-factor or judgemental adjustment of forecasts may create more problems than it solves. In particular, we show how such a practice can lead to the possibility of self-fulfilling fluctuations. To be clear, we examine the extreme case where the judgement variable is not intrinsically related to economic fundamentals at all. Thus our results come from a situation where the forecasting judgement being added is, fundamentally speaking, not useful in forecasting the variables of interest.

We study systems with well-defined rational expectations equilibria. We replace rational expectations with adaptive learning using the methodology of Evans and Honkapohja (2001). We then investigate the equilibrium dynamics of the system if the econometric models of the agents are supplemented with judgement. To define an exuberance equilibrium, we first require that the perceived evolution of the economy corresponds to the actual evolution by imposing a consistent expectations equilibrium concept on the model as developed by Hommes and Sorger (1998) and Hommes, Sorger, and Wagener (2004). Under this requirement, the autocovariance generating functions of the perceived and actual laws of motion correspond exactly. Secondly, we require individual rationality in individual agents’ choice to include the judgement variable in their forecasting model, given that all other agents are using the judgement variable and hence causing it to influence the actual dynamics of the macroeconomy. Finally, we require learnability or expectational stability. When all three of these requirements are met, we say that an exuberance equilibrium exists. In our exuberance equilibria, all agents would be better off if the judgement variable were not being used, but as it is being used, no agent wishes to discontinue its use. We view this as a Nash
equilibrium in beliefs.

1.3 Near-rationality

Though the use of judgement has been justified as a Nash equilibrium, our equilibrium does not correspond exactly to a rational expectations equilibrium. This is because the judgement variable is assumed to be unavailable in the statistical part of the forecasting. We think of this as reflecting the separation of the econometric forecasting unit from the actual decision makers. Our focus is mainly on private decision makers, who have the econometric forecast as an input to which they are free to add the judgement variable. The judgementally adjusted forecasts are the basis for the decisions and actions of the agents, but the adjustments are not directly observable.

In other words, we are assuming that the judgement variable is not one that can be extracted by the econometric forecasting unit and converted into a statistical time series that can formally be utilized in an econometric forecasting model. In a similar vein the decision makers face a dichotomy in their use of judgement: they either incorporate the variable as an add-factor or they ignore it and directly use the econometric forecast. This inability of the decision makers to transmit to the econometric forecasters in a quantitative way the judgement aspects behind their final economic decisions is the source of the deviation from full rational expectations and the reason for our use of the term “near-rationality.”

1.4 Main findings

We isolate conditions under which exuberance equilibria exist in widely-studied dynamic frameworks in which the state of the system depends on

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2 The term “near rationality” has been used elsewhere in the literature, often to mean less-than-full maximization of utility. See, for example, Akerlof and Yellen (1985) and Caballero (1995). Ball (2000) analyzes a model where the agents use a forecasting model that does not encompass the equilibrium law of motion—a “restricted perception.” Our concept is based on full optimization but subject to the restriction that some information is not quantifiable—“judgement.”
expectations of future endogenous variables. We study two applications of a general linear model, a simple univariate asset-pricing model as well as the canonical New Keynesian model of Woodford (2003) and Clarida, Gali, and Gertler (1999). We interpret the exuberance equilibrium in the asset-pricing model as an example of “excess volatility.” In the New Keynesian application, the exuberance equilibria can also exhibit considerable volatility relative to the underlying fundamental rational expectations equilibrium in which judgement does not play a role.

Our results may lead one to view the possibility of exuberance equilibria as particularly worrisome, as exuberance equilibria may exist even in otherwise benign circumstances. In particular, we show that exuberance is a clear possibility even in the case where the underlying rational expectations equilibrium is unique (a.k.a. determinate). Thus an interesting and novel finding is the possibility of “sunspot-like” equilibria, but without requiring that the underlying rational expectations equilibrium of the model is indeterminate.\(^3\) In a sense, we find “sunspot-like” equilibria without indeterminacy.

In the policy-oriented New Keynesian application, our findings suggest a new danger for policy makers: Choosing policy to induce determinacy and learnability may not be enough, because the policy maker must also avoid the prospect of exuberance equilibria.\(^4\) We show how policy may be designed to avoid this danger. More specifically, in the cases we study policy that is more aggressive than the requirements for determinacy and learnability is needed to avoid the possibility of exuberance equilibria.

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\(^3\)Indeterminacy and sunspot equilibria are distinct concepts, as discussed in Benhabib and Farmer (1999). We consider only linear models, for which the existence of stationary sunspot equilibria requires indeterminacy—see for example Propositions 2 and 3 of Chiappori and Guesnerie (1991).

\(^4\)For discussions of determinacy and learnability as desiderata for the evaluation of monetary policy rules, see Bullard and Mitra (2002) and Evans and Honkapohja (2003a). For a survey see Evans and Honkapohja (2003b).
2 Economies with judgement

2.1 A general linear model

Our results depend on the idea that agents participating in macroeconomic systems are learning using recursive algorithms, and that the systems under learning eventually converge. In many cases, as discussed in Evans and Honkapohja (2001), this convergence would be to a rational expectations equilibrium. The crucial aspect for the present paper is that once agents have their macroeconometric forecast from their regression model, the forecast is then judgementally adjusted.

To fix ideas, consider an economy which may be described by

\[ y_t = \beta y_{t+1}^e + u_t \]  

where \( y_t \) is a vector of the economy’s state variables, \( \beta \) is a conformable matrix of parameters, and \( u_t \) is a vector of stochastic noise terms. For convenience we have dropped any constants in this equation. The term \( y_{t+1}^e \) represents the possibly non-rational expectations of the private sector agents.

The judgement vector in the economy follows

\[ (I - \rho L) \xi_t = \eta_t \]  

where \( I \) is a conformable identity matrix, \( \rho \) is a conformable matrix with roots inside the unit circle, \( L \) is a lag operator, \( \xi_t \) is a vector of judgement variables, and \( \eta_t \) is a vector of stochastic noise terms. We assume that \( u_t \) and \( \eta_t \) evolve independently, so that the judgement variables have no fundamental effect on the economy described by equation (1). This is obviously an important and extreme assumption but it is also the one that we think is the most interesting for the purpose of illustrating our main points, as it is the starkest case. It would be interesting to allow for some correlation between \( u_t \) and \( \eta_t \) in future research.

The hallmark of the recursive learning literature is the assignment of a perceived law of motion to the agents, so that we can view them as using recursive algorithms to update their forecasts of the future based on actual
data produced by the system in which they operate.\footnote{We can think of this as corresponding to the existence of a forecasting community using econometric-based models to guide the expectations of private sector and governmental agents. Forecasting communities like this exist in all industrialized nations. Our analysis differs from but is related to the literature in finance on strategic professional forecasting, see e.g. Ottaviani and Sørensen (2004) and the references therein.} A key aspect of this assignment is to keep the perceived law of motion consistent with the actual law of motion of the system, which will be generated by the interaction of equation (1) with the agents’ expectations formation process. With judgement in the model, it will be apparent below that the VARMA(1,1) perceived law of motion

\[ y_t = by_{t-1} + v_t - av_{t-1}, \]  (3)

can be consistent with the actual law of motion. Here $b$ and $a$ are conformable matrices and $v_t$ is a vector of stochastic noise terms. We require that the perceived law of motion be a stationary and invertible process, that is, the eigenvalues of $b$ and $a$ lie inside the unit circle. We can write this as

\[ y_t = \theta (L) v_t, \]  (4)

where

\[ \theta (L) = (I - bL)^{-1} (I - aL). \]

Agents must form expectations using their perceived law of motion, which implies

\[ E_t^* y_{t+1} = by_t - av_t \]

\[ = [b \theta (L) - a] v_t. \]  (5)

In what follows, we will sometimes call (5) the \textit{econometric forecast}. It is based on the econometric model, the perceived law of motion, alone, and is the traditional description of the expectations formation process in the learning literature.

The novel feature of this paper is that we allow judgement to be added to the macroeconometric forecast:

\[ y_{t+1}^e = E_{t}^* y_{t+1} + \xi_t. \]  (6)
Our goal is to understand the implications of this add-factor judgement on the nature of equilibrium in the economy, and on the convergence of the learning algorithm to equilibrium. We stress that if the judgement vector is null, the model corresponds to a version of systems analyzed extensively in Evans and Honkapohja (2001), and that the conditions for convergence to rational expectations equilibrium in that case are well-established.

2.2 The actual law of motion

Since expectations in the economy are being formed via equation (6), and since these expectations affect the evolution of the economy’s state vector through equation (1), we deduce an actual law of motion for this system as

\[
y_t = \beta \left[ b \theta (L) - a \right] v_t + \beta (I - \rho L)^{-1} \eta_t + u_t
\]

\[
= \beta \left[ b \theta (L) - a \right] \theta (L)^{-1} y_t + \beta (I - \rho L)^{-1} \eta_t + u_t
\]

\[
= \beta \left[ b - a \theta (L)^{-1} \right] y_t + \beta (I - \rho L)^{-1} \eta_t + u_t.
\]

We can then write

\[
\{ I - \beta \left[ b - a \theta (L)^{-1} \right] \} y_t = \beta (I - \rho L)^{-1} \eta_t + u_t.
\]

If we define \( M (L) \equiv I - \beta \left[ b - a \theta (L)^{-1} \right] \), it follows that

\[
y_t = M (L)^{-1} \beta (I - \rho L)^{-1} \eta_t + M (L)^{-1} u_t. \tag{7}
\]

Judgement naturally influences the evolution of the state vector because it influences the views of economic actors concerning the future. The critical question is then whether there are conditions under which the agents would continue to use the add-factored forecast (6) when the economy is evolving according to equation (7). That is, could the agents come to perceive that the judgement vector is in fact useful in forecasting the state vector, even though by construction there is no fundamental relationship? We now turn to this question.


2.3 Consistent expectations

To analyze the effects of judgemental adjustment, we employ the concept of consistent expectations equilibrium as defined by Hommes and Sorger (1998) and extended by Hommes, Sorger, and Wagener (2004). Their core idea is that the econometric forecasters should see no difference between their perceived law of motion for how the economy evolves and the actual data from the economy. One way to develop conditions under which such an outcome may occur is to require that the autocovariance generating function of the perceived law of motion corresponds exactly to the autocovariance generating function of the actual law of motion. For the perceived law of motion, or PLM, equation (4), we can write the autocovariance generating function as

\[ G_{PLM}(z) = \theta(z) \Sigma_v \theta(z^{-1})^T, \]

where \( \Sigma_v \) is the variance-covariance matrix associated with \( v \),

\[ \theta(z) = (I - bz)^{-1}(I - az), \]
\[ \theta(z^{-1}) = (I - bz^{-1})^{-1}(I - az^{-1}), \]

\( z \) is a complex scalar, and \( T \) indicates transpose. For the actual law of motion, or ALM, the autocovariance generating function is the sum of two such functions

\[ G_{ALM}(z) = G_{\eta}(z) + G_{u}(z) \]

by the independence of \( \eta \) and \( u \). Defining \( N(L) \equiv M(L)^{-1} \beta(I - \rho L)^{-1} \) as the term that modifies \( \eta_t \) in equation (7), it follows that

\[ G_{\eta}(z) = N(z) \Sigma_{\eta} N(z^{-1})^T \]
\[ G_{u}(z) = M(z)^{-1} \Sigma_{u} \left[ M(z^{-1})^{-1} \right]^T \]

where \( \Sigma_{\eta} \) and \( \Sigma_{u} \) are the variance-covariance matrices associated with \( \eta \) and \( u \), \( M(z) \equiv I - \beta \left[ b - a\theta(z)^{-1} \right] \), and \( N(z) \equiv M(z)^{-1} \beta(I - \rho z)^{-1} \).

A consistent expectations equilibrium is then characterized by $G_{PLM}(z) = G_{ALM}(z)$, or

$$
\theta(z) \Sigma_{\omega} \theta(z^{-1})^T = N(z) \Sigma_{\eta} N(z^{-1})^T + M(z)^{-1} \Sigma_u \left[ M(z^{-1})^{-1} \right]^T.
$$

This condition can be rewritten as

$$
[(I - \beta b) \theta(z) - a] \Sigma_{\omega} \left[ \theta(z^{-1})^T (I - \beta b)^T - a^T \right] = \\
\beta (I - \rho z)^{-1} \Sigma_{\eta} \left( (I - \rho z^{-1})^{-1} \right)^T \beta^T + \Sigma_u. \tag{10}
$$

We will use this condition in the remainder of the paper, in order to isolate conditions under which a consistent expectations equilibrium, or CEE, exists. We will verify the existence of a solution to the equation (10) analytically for the univariate case.

The formulation just presented defines an exact notion of a consistent expectations equilibrium. In the applications we will also make use of an extended notion of an *approximate* consistent expectations equilibrium. We employ the approximate CEE numerically in the multivariate case, for which (10) is difficult to solve. In the latter concept econometricians use a PLM that is an approximation to but not fully consistent with the ALM in the sense that they do not exactly match all of their autocovariances. However, for an approximate consistent expectations equilibrium we do require that the equilibrium PLM is the best choice within the restricted class considered.

For example, if the econometricians use a VAR(1) forecasting model, the parameters used must provide the best linear projections of the variables being forecast.

### 2.4 Incentives to include judgement

When all agents in the model are making use of the judgementally adjusted forecast described in equation (6), they induce an actual law of motion for the system which is described by equation (7). An individual agent may nevertheless decide that it is possible to make more efficient forecasts by
simply ignoring the judgemental adjustment. If this is possible, then it is not
individually rational for all agents to use the add-factored forecast. We check
this individual forecast efficiency condition by comparing the variance of the
forecast error for the judgemental forecast (6) to the variance of the forecast
error with judgement not included, the econometric forecast (5), under the
condition that all other agents are using the judgementally adjusted forecast
and thus are inducing the actual law of motion (7). This can be written as
follows. Given that
\[ y_{t+1} = M(L)^{-1} \beta (I - \rho L)^{-1} \eta_{t+1} + M(L)^{-1} u_{t+1}, \]
under what conditions is the covariance matrix of \( y_{t+1} - y_{t+1} \) in some
sense smaller than the covariance matrix of \( E^*_i y_{t+1} - y_{t+1} \)?\(^7\) We denote the covari-
ance matrix without judgement as \( \mathcal{M}(0) \) and with judgement as \( \mathcal{M}(1) \). For
most of the paper, we will interpret this condition to mean that the com-
ponent by component comparison of the matrices along the diagonal are all
smaller. That is
\[ \mathcal{M}(0)_i - \mathcal{M}(1)_i > 0 \] (11)
for all diagonal components \( i \). In some parts of the paper we discuss a stricter
requirement which is that \( \mathcal{M}(0) - \mathcal{M}(1) \) is a positive definite matrix. We
regard this version as perhaps too strict to be interesting, but we sometimes
mention it. We will show below that these requirements can be met under
admissible conditions on model parameters.

### 2.5 Learnability

Since we have made an assumption that the econometricians in the model
are learning using recursive algorithms, we also need to impose learnability
of any proposed equilibrium as a condition for plausibility. The econometric
forecasting is based on a time series model which in equilibrium has coef-

\(^7\)We implicitly are assuming that \( y_{t+1} \) and \( E^*_i y_{t+1} \) have the same mean as \( y_{t+1} \), so that
variance of the forecast error is the same as the mean squared error. This will always hold
in our analysis.
mean that the coefficients of the econometric forecasting models are locally stable outcomes of recursive least squares learning. We discuss learnability conditions for more specific contexts in detail later in the paper.

2.6 Exuberance equilibrium

An exuberance equilibrium can now be defined. Given the model with judgement (1), (2), (5), and (6) an \textit{exuberance equilibrium} exists if

1. A consistent expectations equilibrium exists,

2. Individual agents rationally decide to include the (non-trivial) judgement vector in their forecasts by checking that condition (11) is met, and

3. The consistent expectations equilibrium is learnable in the sense of Evans and Honkapohja (2001).

While these requirements define an exuberance equilibrium, we sometimes refer to versions of this definition as a method of categorizing our results. If the individual rationality condition is met in the sense that the difference between the two covariance matrices is a positive definite matrix, in conjunction with the other two requirements, we say that a \textit{strong exuberance equilibrium} exists. Also, the individual rationality condition may be such that some diagonal components of the difference between the two covariance matrices are positive, while others are negative, when all other conditions are met. This means that the agents may or may not come to the conclusion that including the judgemental adjustment is valuable. We will refer to this case as \textit{indefinite}. Another possibility is that the diagonal components of the difference between the two covariance matrices are all negative when all other conditions are met. In this case the agents would most likely conclude that the inclusion of judgement was not valuable. We call this case one of \textit{non-exuberance}. Finally, to be complete, the difference could be a negative definite matrix in which case we say that there is \textit{strong non-exuberance}. 

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In the next section we obtain conditions under which exuberance equilibria exist in the scalar model.

3 Exuberance equilibria in the scalar model

3.1 Asset pricing example

A simple univariate example of the framework (1) is given by the standard present value model of asset pricing. A convenient way of obtaining the key structural equation can be based on the quadratic heterogenous agent model of Brock and Hommes (1998). In their framework agents are myopic mean-variance maximizers who choose the quantity of riskless and risky assets in their portfolio to maximize expected value of a quadratic utility function of end of period wealth.

We modify their framework to allow for shocks to the supply of the risky asset. For convenience we assume homogenous expectations and constant dividends. The temporary equilibrium is given by

\[ p_{t+1}^e + d - R_fp_t = s_t, \]

where \( d \) is the dividend, \( p_t \) is the price of the asset and \( R_f > 1 \) is the rate of return factor on the riskless asset. Here \( s_t \) is a linear function of the random supply of the risky asset per investor, assumed i.i.d. for simplicity.\(^8\)

For convenience we are assuming that the dividend \( d \) is constant and known. Defining \( y_t = p_t - \bar{p} \), where \( \bar{s} = Es_t \) and \( \bar{p} = (d - \bar{s})/(R_f - 1) \), we obtain a scalar version of (1) with \( \beta = R_f^{-1} \) and \( u_t = -R_f^{-1}(s_t - \bar{s}) \). Note that 0 < \( \beta < 1 \).

The univariate equation (1) is a benchmark model of asset pricing and there are, of course, alternative ways to derive the same equation. Because 0 < \( \beta < 1 \) the model is said to be regular or determinate, that is, under rational expectations there is a unique nonexplosive solution, given by the

\(^8\)Using the notation of Brock and Hommes (1998) \( s_t = a\sigma^2z_{st} \), where \( \sigma^2 \) is the conditional variance of excess returns (assumed constant) and \( a \) is a parameter of the utility function.
“fundamentals” solution $y_t = u_t$. In particular, under rational expectations, sunspot solutions do not exist.

### 3.2 The general scalar model

Are there conditions under which an exuberance equilibrium could exist? There are, and we argue that the conditions are in fact worrisomely plausible. In order to obtain some intuition, we turn first to the analysis of the general scalar case.

We maintain the same notation as in Section 2.1, but we let $y_t$, $\xi_t$, $u_t$, $\eta_t$, and $v_t$ be scalar-valued, and we proceed with the understanding that $\beta$, $\rho$, $a$, and $b$ are scalars, and that $I = 1$ in the scalar case. We assume that $0 < \beta < 1$ as in the asset pricing example. We also make the assumption that the exuberance variable is positively serially correlated, so that $0 < \rho < 1$.

The econometric forecast can be written as

$$E_t^* y_{t+1} = b y_t - a v_t$$

$$= b \theta (L) v_t - a v_t$$

$$= \left( \frac{b - a}{1 - bL} \right) v_t.$$  \hspace{1cm} (12)

The add-factored forecast is

$$y^e_{t+1} = E_t^* y_{t+1} + \xi_t.$$  \hspace{1cm} (13)

The latter forecast induces an actual law of motion for the economy

$$y_t = \beta y^e_{t+1} + u_t$$

$$= \beta \left( \frac{b - a}{1 - bL} \right) v_t + \frac{\beta}{1 - \rho L} \eta_t + u_t$$

$$= \beta \left( \frac{b - a}{1 - bL} \right) \left( \frac{1 - bL}{1 - aL} \right) y_t + \frac{\beta}{1 - \rho L} \eta_t + u_t.$$  \hspace{1cm} (14)

Solving for $y_t$ implies that the actual law of motion is

$$y_t = \frac{1 - aL}{\beta (a - b) + 1 - aL} \left( \frac{\beta}{1 - \rho L} \eta_t + u_t \right).$$

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3.3 Conditions for consistent expectations

To apply the idea of consistent expectations equilibrium, we follow Section 2.3 and require that the autocovariance generating functions of the perceived and actual laws of motion are the same. The autocovariance generating function for the perceived law of motion in the scalar case is given by

\[ G_{PLM}(z) = \sigma^2 v (1 - az)(1 - az^{-1}) (1 - bz)(1 - bz^{-1}). \]

(15)

For a consistent expectations equilibrium we require that \( G_{ALM}(z) = G_{PLM}(z) \).

We note that this implies an ALM that takes an ARMA(1,1) form with identical coefficients to those of the PLM.\(^9\)

For the actual law of motion,

\[ G_{ALM}(z) = G_\eta(z) + G_u(z) \]

where

\[ G_\eta(z) = \frac{\sigma^2 \beta^2 (1 - az)(1 - az^{-1})}{[\beta (a-b) + 1 - az][\beta (a-b) + 1 - az^{-1}](1 - \rho z)(1 - \rho z^{-1})}, \]

and

\[ G_u(z) = \frac{\sigma^2 (1 - az)(1 - az^{-1})}{[\beta (a-b) + 1 - az][\beta (a-b) + 1 - az^{-1}]} \]

We use these functions to demonstrate the following result in Appendix A.

Lemma 1 There exists a consistent expectations equilibrium with \( b = \rho \) and \( a \in [0, \rho] \).

As also shown in Appendix A, there are interesting limiting cases: when \( \sigma^2_\eta \to 0 \), so that the relative variance of the judgement process is small, \( a \to \rho \), while for \( \sigma^2_u \to 0 \), meaning that the relative variance of the fundamental process is small, \( a \to 0 \). Thus the value of \( a \) depends in an interesting way on the relative innovation variance \( R \equiv \sigma^2_\eta/\sigma^2_u \), as well as the discount factor \( \beta \) and the serial correlation \( \rho \). Since a value of \( a \in [0, \rho] \) always exists, the

\(^9\)See Brockwell and Davis (1991, p. 90, Remark).
conditions for a consistent expectations equilibrium can always be met in the scalar case. Appendix A also makes clear that there is a second, negative value of $a$ which equates the two autocovariance generating functions. We found that the other conditions for exuberance equilibrium are not met at this value of $a$, and we refer to it only in passing in the remainder of the paper.

We now ask whether individual rationality holds with respect to inclusion of the judgement variable in making forecasts.

### 3.4 Incentives to include judgement in the scalar case

In this section, we use the condition from the consistent expectations calculation that $b = \rho$. We then note that $v_t = (\frac{1 - \rho L}{1 - a L}) y_t$. The econometric forecast is therefore given by

$$E_t^* y_{t+1} = \frac{\rho - a}{1 - \rho L} v_t = \frac{\rho - a}{1 - a L} y_t$$

whereas the judgementally adjusted forecast is given by

$$y_{t+1}^e = \frac{\rho - a}{1 - a L} y_t + \frac{1}{1 - \rho L} \eta_t.$$  

The question from an (atomistic) individual agent’s point of view is then whether they should use (16) or (17) as a basis for their expectations of the future state of the economy. We assume for the purposes of this calculation that all other agents in the economy are making use of the judgementally adjusted forecast.

Is it possible for the variance of the judgementally adjusted forecast to be lower than the variance of the econometric forecast? It is. Consider the special case when $\sigma^2_\eta \to 0$ so that the positive root $a \to \rho$. Then it is shown in Appendix B that

$$FE_{NJ} = \frac{\beta}{1 - \rho L} \eta_{t+1}$$

whereas

$$FE_J = \frac{\beta (1 - \beta^{-1} L)}{1 - \rho L} \eta_{t+1}.$$
The difference between the variances of these two forecast errors is then\textsuperscript{10}

\[ \text{Var}[FE_J] - \text{Var}[FE_{NJ}] = \frac{\beta^{-1}}{1 - \rho^2} (\beta^{-1} - 2\rho). \]

This can be less than zero if and only if

\[ \rho \beta > \frac{1}{2}. \]  

(18)

By continuity, it follows that if \( \beta > 1/2 \) there are non-trivial judgement processes (with \( \rho > 1/2\beta \) and \( \sigma^2_\eta > 0 \) sufficiently small) for which the agents have incentives to include the process as an add factor in their forecasts. The preceding argument considered the limiting case \( a \to \rho \), but as we will show below, it is not necessary for \( a \) to be close to \( \rho \) for our results to hold.

We conclude that individuals will decide to use the judgementally adjusted forecast in cases where \( \rho \) is relatively large, meaning that the serial correlation in the judgement variable is substantial, and when \( \beta \) is simultaneously relatively high, meaning expectations are relatively important in determining the evolution of the economy. We remark that these conditions are exactly the ones that correspond to the most likely scenario for this type of model.

Another, polar opposite, special case is one where \( \sigma^2_u \to 0 \) so that the positive root \( a \to 0 \). Then

\[ FE_{NJ} = \frac{\beta}{1 - \rho \beta} \eta_{t+1} \]

whereas

\[ FE_J = \frac{\beta}{(1 - \rho \beta)} \frac{(1 - \beta^{-1}L)}{(1 - \rho L)} \eta_{t+1}. \]

The difference between the variances of these two forecast errors is then

\[ \text{Var}[FE_J] - \text{Var}[FE_{NJ}] = \frac{(\beta^{-1} - \rho)^2}{1 - \rho^2}. \]

---

\textsuperscript{10}See, for instance, Harvey (1981, p. 40). The variance of \( x_t = [(1 + \theta L) / (1 - \phi L)] \varepsilon_t \) is \( \left( (1 + \theta^2 + 2\phi \theta) / (1 - \phi^2) \right) \sigma^2_\varepsilon \).
This can never be less than zero given maintained assumptions. We conclude that it cannot be individually rational for agents to use a judgementally adjusted forecast in the scalar case when the relative variance of the judgemental variable is very large.

By continuity we deduce from these two special cases that there are values of $R = \frac{\sigma^2_n}{\sigma^2_u} \in (0, \infty)$ such that $a \in (0, \rho)$ and agents rationally choose to use a judgementally adjusted forecast, given that all other agents are doing so.\footnote{The case with $a \to \rho$ is a near-common factor representation of the time series, but the required variances remain continuous in the parameters, as can be seen from the formulae in Appendix B.}

The general case involves the variance of an ARMA(2,2) process. We show how to compute this variance in Appendix B, and illustrate the findings in Figure 1.

The figure is drawn for $\beta = .9$ and $\rho = .9$, which corresponds to what might be regarded as a realistic case. The variances of the forecast errors with and without judgement are plotted on the vertical axis, while the value of $a$ is plotted on the horizontal axis. Each value of $a$ between zero and $\rho$ corresponds to a different relative variance $R = \frac{\sigma^2_n}{\sigma^2_u}$, and larger values of $R$ are associated with smaller values of $a$. We have already seen from the examination of special cases that as $R \to \infty$, $a \to 0$ and we expect the forecast error variance of the econometric forecast to be smaller. This result is borne out in the figure. In addition, we expect the variance of the judgementally adjusted forecast to be lower when $R \to 0$, in which case $a \to \rho$. This is also borne out in the figure. But the figure also shows intermediate cases, and indicates that $a$ does not have to be particularly close to $\rho$ for the individual rationality condition to be met. In fact, the two forecast error variances are equal at $a \approx .21$, which is far from the value of $\rho$ in this example, which is .9. We conclude that the conditions for exuberance equilibria to exist are quite likely to be met for a wide range of relative variances $R$ provided both $\beta$ and $\rho$ are relatively close to one.

This intuition can be partially verified by checking cases where $\beta$ and $\rho$
Figure 1: The variance of the forecast error, with (FEJ) and without (FENJ) judgement. The variance can be lower with judgement included, even for values of $a$ far from $\rho$.

are not so large. Based on condition (18), one might conjecture that the individual rationality constraint is binding at values $\rho \beta < 1/2$. In fact, at $\rho = .7$ and $\beta = .7$, an exercise like the one behind Figure 1 shows that there are no values of $a$ that make the judgementally adjusted forecast preferable to the econometric forecast.

3.5 Learnability in the scalar case

We study the stability of the system under learning following the literature on least squares learning in which the economic agents making forecasts are assumed to employ econometric models with parameters updated over time.
as new data becomes available.\textsuperscript{12} The standard way to analyze systems under learning is to employ results on recursive algorithms such as recursive least squares. In many applications it can be shown that there is convergence to rational expectations equilibrium, provided the equilibrium satisfies a stability condition.

In the current context, the consistent expectations equilibrium formulated above takes the form of an ARMA(1,1) process. Estimation of ARMA(1,1) processes is usually done using maximum likelihood techniques, taking us beyond standard least squares estimation. Recursive maximum likelihood (RML) algorithms are available and they have formal similarities to recursive least squares estimation. Because this technical analysis is relatively unfamiliar, we confine the formal details to Appendix C. However, the results are easily summarized. Let $a_t$ and $b_t$ denote estimates at time $t$ of the coefficients of ARMA forecast function (12). Numerical computations using RML indicate convergence of $(a_t, b_t)$ to $(a, \rho)$, where $a > 0$ is the CEE value given in Lemma 1. Thus this CEE is indeed stable under learning. Moreover, in Section 3.6 we state a formal convergence result as part of our existence theorem.

It is also of interest to relax the requirement that the econometricians learn an exact CEE and to study approximate CEE. There are two related reasons for our interest in the approximate CEE. In the multivariate setting it is much more difficult to verify existence using our method based on autocovariance generating functions. However, it is numerically straightforward to use a learning algorithm to calculate approximate CEE taking the form of a VAR($p$) process. Secondly, in multivariate settings VARMA procedures are not widely used and the standard forecasting tool in practice is to estimate VAR processes. It will therefore be natural to focus on approximate CEE taking the VAR form. To prepare for this we here look at approximate CEE in the univariate setting. When the approximate CEE satisfy the incentives

\textsuperscript{12}Evans and Honkapohja (2001) is a systematic treatment of adaptive learning and its implications in macroeconomics. Evans and Honkapohja (1999), Marimon (1997) and Sargent (1993, 1999) provide surveys of the field.
to include judgement and are stable under learning, then we refer to them as *approximate exuberance equilibria*.

Consider the case in which the econometricians have AR(\(p\)) perceived laws of motion. These PLMs do not lead to an exact CEE but for \(p\) large enough the fixed points of learning will be good approximations to the corresponding exact CEE. The PLM is specified as

\[
y_t = \sum_{i=1}^{p} b_i y_{t-i} + v_t. \tag{19}\]

This leads to forecasts with judgement

\[
y_{t+1}^e = \sum_{i=0}^{p-1} b_{i+1} y_{t-i} + \xi_t,
\]

where \(\xi_t\) is the judgement term, and hence to the ALM

\[
y_t = (1 - \beta b_1)^{-1} \left\{ \sum_{i=1}^{p-1} \beta b_{i+1} y_{t-i} + \beta \xi_t + u_t \right\}. \tag{20}\]

It is easily verified that the ALM is an ARMA(\(p,1\)) process and this is the way in which the AR(\(p\)) PLM can only give an approximate CEE.

Let \(b = (b_1, \ldots, b_p)\) and let \(P[y_t | Y_{t-1}] = T(b) Y_{t-1}\) be the linear projection of \(y_t\) on \(Y_{t-1}\) where \(Y'_{t-1} = (y_{t-1}, \ldots, y_{t-p})\). Using standard results on linear projections,

\[
T(b) = (Ey_t Y'_{t-1}) (EY_{t-1} Y'_{t-1})^{-1}. \tag{21}\]

An approximate CEE \(\tilde{b}\) satisfies the equation \(\tilde{b} = T(\tilde{b})\). To compute \(T(b)\) one can write the system in first order form

\[
z_t = B z_{t-1} + D \begin{pmatrix} u_t \\ \eta_t \end{pmatrix}
\]

with \(z_t = (Y'_t, \xi_t)'\). The relevant values for \((Ey_t Y'_{t-1})\) and \((EY_{t-1} Y'_{t-1})\) can be obtained from the equation

\[
vec(Var(z_t)) = [I - B \otimes B]^{-1} vec(D \left[ Var \begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \right]) D'.
\]

20
Here $\text{vec}(K)$ is the vectorization of a matrix $K$ and $\otimes$ is the Kronecker product. The equilibrium $\bar{b}$ can then be calculated by the $E$-stability algorithm

$$b_s = b_{s-1} + \gamma(T(b_{s-1}) - b_{s-1}),$$

(22)

where $\gamma$ is chosen to be a small positive constant.

This procedure will automatically give us learnable equilibrium in the following sense. The econometricians are estimating an AR($p$) PLM for $y_t$ and are assumed to update their parameter estimates over time using recursive least squares (RLS). As previously explained, the decision makers add their judgement adjustment to the econometricians’ forecast and, together with the variable $u_t$, the current value of $y_t$ is determined. The vector $T(b)$ denotes the true coefficients projection for a given forecast coefficients $b$. Under RLS learning it can be shown that econometrician’s estimates $b_t$ at time $t$ on average move in the direction $T(b_t)$. Equation (22) describes this adjustment in notional time $s$. Using the techniques of Evans and Honkapohja (2001), it can be shown that RLS learning converges locally to $\bar{b}$ if it is a locally asymptotically stable fixed point (22) for sufficiently small $\gamma > 0$.

To illustrate this procedure we have computed AR(3) approximate CEE for $\beta = 0.9$, $\rho = 0.7$ and $R = 1$. The exact ARMA(1,1) CEE given in Lemma 1 is $b = 0.7$ and $a = 0.180814$. Directly computing the approximate AR(3) CEE we obtain the solution

$$b_1 = 0.517259, b_2 = 0.0929807, b_3 = 0.0157536.$$

These are reasonable approximations to the first three terms of the series expansion of $(1 - bz)/(1 - az)$. These results show that AR(3) approximate CEE associated with this ARMA(1,1) CEE is stable under least squares learning.

3.6 Existence and properties of equilibrium

We now return to exact CEE, taking the ARMA(1,1) form, and collect the various results above. The following theorem gives the key results about exis-
tence of an exuberance equilibrium in the univariate model and characterizes its asymptotic variance:

**Theorem 2** Consider the univariate model with judgement and suppose that \( \beta > 1/2 \). Then

(i) for appropriate AR(1) judgement processes there exists an exuberance equilibrium and

(ii) the exuberance equilibrium has a higher asymptotic variance than the rational expectations equilibrium.

**Proof.** (i) The preceding analysis has verified that the conditions (1)-(2) for an exuberance equilibrium in Section 2.6 are met for all \( \sigma^2_y > 0 \) sufficiently small. In Appendix C it is proved that condition (3) also holds, that is, the CEE is stable under RML learning, when \( \sigma^2_y > 0 \) is sufficiently small.

(ii) The rational expectations equilibrium for the univariate model is \( y_t = u_t \) since \( 0 < \beta < 1 \) and \( u_t \) is iid with mean zero. The exuberance equilibrium with \( a > 0 \) can be represented as the ARMA(1,1) process \( y_t = \rho y_{t-1} + v_t - av_{t-1} \) where \( a \) solves the equation (31) given in Appendix A. From (30) it is seen that

\[
\sigma^2_v = \frac{\rho}{a(\beta(a - \rho) + 1)} \sigma^2_u > \sigma^2_u
\]

as \( a < \rho \) and \( 0 < \beta, \rho < 1 \). Next, using the formula for the variance of an ARMA(1,1) process we have

\[
\sigma^2_y = \frac{1 + a^2 - 2\rho a}{1 - \rho^2} \sigma^2_v
\]

and since \( \frac{1 + a^2 - 2\rho a}{1 - \rho^2} > 1 \), the result follows. ■

The theorem states that in an exuberance equilibrium, we expect the variance of the state variable \( y_t \) to be larger than it would be in a fundamental rational expectations equilibrium. This is because the REE has \( y_t = u_t \), so that \( \sigma^2_y = \sigma^2_u \), but in an exuberance equilibrium \( \sigma^2_y > \sigma^2_u \). A natural question is whether this effect is economically meaningful, or if exuberance conditions are only met for situations in which the variance \( \sigma^2_y \) is just trivially larger.
Table 1: Excess Volatility

<table>
<thead>
<tr>
<th>σξ/σu</th>
<th>β = 0.95, ρ = 0.70</th>
<th>β = 0.95, ρ = 0.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.44 0.24 − − −</td>
<td>0.45 0.24 0.15 0.10 −</td>
</tr>
<tr>
<td>σy/σu</td>
<td>1.54 2.74 − − −</td>
<td>2.70 5.82 9.11 12.43 −</td>
</tr>
</tbody>
</table>

Table 1: Exuberance equilibria in the asset pricing model. A dash indicates that exuberance equilibrium does not exist. The second calculated line of each block gives one measure of the degree of excess volatility generated. The model can easily generate substantial excess volatility.

than the fundamental variance. This is not clear from the theorem since $a$ is itself a nonlinear function of $β$, $ρ$, and $R = σ^2_η/σ^2_u$. It is also of interest to know if the excess volatility effect isolated in the theorem is large enough to be comparable to empirical estimates of the degree of excess volatility in financial data. One famous calculation due to Shiller (1981) put the ratio of the standard deviation of U.S. stock prices to fundamental factors at between 5 and 13.

Table 1 provides some illustrative calculations of exuberance equilibria for representative parameter values. In the table, instead of considering the relative variance $R = σ^2_η/σ^2_u$, we consider the perhaps more intuitive ratio of the standard deviation of the exuberance variable to the standard deviation of the fundamental shock $σξ/σu$.13 Ratios of $σξ/σu$ near unity correspond to ratios of innovation variances $σ^2_η/σ^2_u$ on the order of 0.1 for a high degree of serial correlation, so that the noise associated with judgement in the economy is actually quite modest. Each block in the table gives results for several possible values of $σξ/σu$, ranging from 0.5 to 2.5. We examine the empirically realistic case where the discount factor $β$ is slightly less than one,

13Note that $σ^2_ξ = σ^2_η/(1 − ρ^2)$.
and where the degree of serial correlation $\rho$ is substantial. A dash in the table indicates that exuberance equilibrium does not exist for the indicated parameter values. The “$a$” row of each block gives the value of $a$ associated with the indicated values of $\beta$, $\rho$, and $\sigma_\xi/\sigma_u$. The next row of each block gives a measure of excess volatility corresponding to Shiller’s (1981) concept, namely, $\sigma_y/\sigma_u$. The results indicate that these measures are often in the range of 5 to 13 estimated by Shiller. We conclude based on this illustrative calculation that the model can generate substantial excess volatility without difficulty. We remark that if we push the discount factor $\beta$ closer to unity, the degree of excess volatility can rise to very high levels for high degrees of serial correlation, with $\sigma_y$ many hundreds of times larger than $\sigma_u$. In this sense, the model can generate arbitrarily large amounts of excess volatility.

We emphasize that $0 < \beta < 1$ corresponds to the determinate case for this model, that is, the rational expectations equilibrium is unique. However, for $0.5 < \beta < 1$ exuberance equilibria exist even though sunspot equilibria do not exist. We think this feature of our findings is striking as it means that what would normally be regarded as benign circumstances can actually be dangerous situations for the possibility of near-rational exuberance.

The theorem shows that the basic asset pricing model is consistent with excess volatility. If investors incorporate judgemental factors that are strongly serially correlated they will find that this improves their forecasts, but in an exuberance equilibrium this will generate significant stationary asset price movements in excess of those associated with fundamental factors. The stationarity of our exuberance movements is in marked contrast to the literature on rational asset price bubbles. Because the latter are explosive in the basic model, the literature on rational bubbles has been punctuated by controversy and complicated by the need to construct valid tests for non-stationary bubbles.
4 A monetary policy example

4.1 New Keynesian macroeconomics

We now wish to study exuberance equilibria in a simple microfounded macroeconomic model suggested by Woodford (2003) and Clarida, Gali, and Gertler (1999). We use a simple, three-equation version given by

\begin{align*}
x_t &= x_{t+1}^{e} - \sigma^{-1} \left[ r_t - \pi_t^{e} \right] + \tilde{u}_{x,t}, \quad (23) \\
\pi_t &= \kappa x_t + \delta \pi_{t+1}^{e} + \tilde{u}_{\pi,t}, \quad (24) \\
r_t &= \varphi_{\pi} \pi_t + \varphi_{x} x_t. \quad (25)
\end{align*}

In these equations, \(x_t\) is the output gap, \(\pi_t\) is the deviation of inflation from target, and \(r_t\) is the deviation of the nominal interest rate from the value that is consistent with inflation at target and output at potential. All variables are expressed in percentage point terms and the steady state is normalized to zero. The terms \(\tilde{u}_{x,t}\) and \(\tilde{u}_{\pi,t}\) represent stochastic disturbances to the economy. The parameter \(\sigma^{-1}\) is related to the elasticity of intertemporal substitution in consumption of a representative household. The parameter \(\kappa\) is related to the degree of price stickiness in the economy, and \(\delta\) is the discount factor of a representative household.\(^{14}\) The third equation describes the Taylor-type policy rule in use by the policy authority, in which the parameters \(\varphi_{\pi}\) and \(\varphi_{x}\) are assumed to be positive. In the formulation (23)-(25), only private sector expectations affect the economy.

Substituting (25) into (23) and writing the system in matrix form gives

\[ y_t = \beta y_{t+1}^{e} + u_t \quad (26) \]

where \(y_t = [x_t, \pi_t]'\), \(y_{t+1}^{e} = [x_{t+1}^{e}, \pi_{t+1}^{e}]'\), \(u_t = C\tilde{u}_t\), \(\tilde{u}_t = [\tilde{u}_{x,t}, \tilde{u}_{\pi,t}]'\) with covariance matrix

\[ \Sigma_u = \begin{bmatrix} \sigma_{u,11}^2 & \sigma_{u,12}^2 \\ \sigma_{u,21}^2 & \sigma_{u,22}^2 \end{bmatrix}, \]

\(^{14}\)This formulation of the model is based upon individual Euler equations under (identical) private sector expectations. Other models of bounded rationality are possible, see, for instance, Preston (2003) for a formulation in which long-horizon expectations directly affect individual behavior.
\[
\beta = \frac{1}{\sigma + \varphi_x + \kappa \varphi_x} \begin{bmatrix} \sigma & 1 - \delta \varphi_x \\ \kappa \sigma & \kappa + \delta (\sigma + \varphi_x) \end{bmatrix},
\]
and
\[
C = \frac{1}{\sigma + \varphi_x + \kappa \varphi_x} \begin{bmatrix} \sigma & -\varphi_x \\ \kappa \sigma & \sigma + \varphi_x \end{bmatrix}.
\]

### 4.2 Approximate exuberance equilibria

For the reasons given in Section 3.5 we focus on approximate exuberance equilibria based upon a VAR\((p)\) perceived law of motion. This is a straightforward multivariate extension of the approach used in the univariate case to compute approximate CEE based on AR\((p)\) PLMs. The PLM is specified as equation (19), again taking the form

\[
y_t = \sum_{i=1}^{p} b_i y_{t-i} + v_t.
\]

However, this now describes a VAR\((p)\) as \(y_t\) and \(v_t\) are now \(2 \times 1\) vectors and the \(b_i\) are \(2 \times 2\) matrices. The actual law of motion is given by equation (20). Note that \(b = (b_1, \ldots, b_p)\) is now a \(2 \times 2p\) matrix and the linear projection is now \(P[y_t | Y_{t-1}] = T(b) Y_{t-1}\), where \(Y_{t-1}' = (y_{t-1}', \ldots, y_{t-p}')\) and \(T(b)\) is given by equation (21). An approximate CEE \(\bar{b}\) satisfies the equation \(\bar{b} = T(\bar{b})\). The equilibrium \(\bar{b}\) can again be calculated numerically by the E-stability algorithm (22). This procedure also guarantees stability of the equilibrium under RLS learning.

With the VAR\((p)\) model the forecasts without and with judgement are given by

\[
y_{t+1}' = \sum_{i=1}^{p} b_i y_{t+1-i} + k \xi_t,
\]
where \(k = 0\) or \(1\). By setting up the model in first order state space form, and including in the state the forecast errors \(y_t - y_t'\) with and without judgement, it is straightforward to compute \(\mathcal{M}(0) - \mathcal{M}(1)\) and test numerically for the existence of exuberance equilibria.
4.3 Results

4.3.1 A Taylor-type monetary policy rule

We now illustrate the possibility of approximate exuberance equilibria in the New Keynesian model. We use Woodford’s (2003) calibration $\sigma = 0.157$, $\kappa = 0.024$, and $\delta = 0.99$. For the exuberance variable we assume the matrix describing the degree of serial correlation is $\rho = \text{diag}(0.99, 0.95)$ and $\Sigma_\eta = \text{diag}(0.0035, 0.0035)$. The variances of the fundamental shocks are assumed to be $\Sigma_\eta = \text{diag}(1.1, 0.03)$. No real attempt has been made to calibrate the shocks except to choose values that, in the exuberance equilibrium, roughly match U.S. inflation and output-gap variances measured in percent.

The policy parameters $\varphi_\pi$ and $\varphi_x$ can be varied and we are interested in values of $\varphi_\pi$ and $\varphi_x$ that might be consistent with exuberance equilibrium. Consider $\varphi_\pi = 1.05$ and $\varphi_x = 0.05$. These values satisfy the Taylor principle and deliver a determinate rational expectations equilibrium in the usual setup; see Bullard and Mitra (2002). Suppose that econometricians estimate a VAR(3). In the approximate CEE the coefficients of the vector autoregression are approximately

$$b_1 = \begin{pmatrix} 0.0975 & -0.3319 \\ 0.0759 & 0.8775 \end{pmatrix},$$

$$b_2 = \begin{pmatrix} 0.0976 & 0.0108 \\ 0.0012 & 0.0902 \end{pmatrix},$$

and

$$b_3 = \begin{pmatrix} 0.0586 & 0.0731 \\ -0.0037 & 0.0071 \end{pmatrix}.$$

This corresponds to a stationary process. The output variance is approximately 2.54 and the inflation variance is approximately 6.14. The matrix describing the key condition for individual rationality is computed to be

$$\mathcal{M}(0) - \mathcal{M}(1) = \begin{pmatrix} 0.0219 & -0.0195 \\ -0.0195 & 0.0209 \end{pmatrix}.$$

This is a positive definite matrix. Hence the CEE is a strong exuberance equilibrium according to the definitions given in Section 2.6. As in the scalar
model the exuberance equilibrium exhibits excess volatility. In fact, the ratio of output-gap standard deviation in the exuberance equilibrium to its standard deviation in the fundamental rational expectations equilibrium is about 1.5 and for inflation standard deviation the ratio is almost 16!

In this example we can also show that a change in the Taylor-rule coefficients can diminish the likelihood of exuberance equilibria. When $\phi_\pi$ is increased to 1.1 the equilibrium is no longer strongly exuberant but it does remain exuberant. However, if $\phi_\pi$ is increased to 1.5 and $\phi_x$ is increased to 0.1, the possibility of an exuberance equilibrium is eliminated. In this sense, a more aggressive policy tends to reduce the likelihood of an exuberance equilibrium.

We next analyze the idea that more aggressive policy is less likely to be associated with the existence of exuberance equilibrium more systematically. For this, we calculate the conditions for exuberance equilibrium using the calibration given above but allowing the Taylor rule coefficients to vary. The results are given in Figure 2, where $\phi_\pi \in (0, 1.25)$ and $\phi_x \in (0, 0.25)$ at selected grid points. The open squares indicate the points where determinacy and learnability of the rational expectations equilibrium hold for this model.\(^{15}\) The figure displays the points at which exuberance equilibria exist. These points tend to be for values of $\phi_x$ less than about 0.08, and for values of $\phi_\pi$ up to 1.25. Again, these exuberance equilibria exist in the region associated with determinacy, and therefore can arise in parameter regions where sunspot equilibria are ruled out.

While Figure 2 illustrates where exuberance equilibria exist in this economy, it is not comforting regarding the possibility that policymakers may be able to choose policy parameters so as to rule out exuberance equilibria. According to the figure, either an exuberance equilibrium exists or the indefinite case arises (plain open boxes in the figure). However, if we expand the space of points considered, it becomes apparent that more aggressive policy can produce situations characterized by non-exuberance. This is shown in

\(^{15}\)The blank area to the left in this figure is associated with indeterminacy of rational expectations equilibrium.
Figure 2: Exuberance equilibria in the New Keynesian model. Open boxes indicate points where the REE is determinate. Triangles indicate points where exuberance equilibria exist.

Figure 3, where the region of the policy parameter space has been expanded so that $\varphi_\pi \in (0, 2.5)$ and $\varphi_x \in (0, 0.45)$ at selected grid points. In this figure, the region associated with exuberance from the previous figure appears near the point $(1, 0)$. However, there is now a region of the policy parameter space that is associated with non-exuberance. This part of the space involves more aggressive reactions to both inflation deviations and the output gap. In this sense, a more aggressive policy can mitigate the possibility of exuberance equilibrium in this economy.
Figure 3: A sufficiently aggressive Taylor-type policy is associated with non-exuberance, denoted by open circles.

4.3.2 A forward-looking monetary policy rule

It is also of interest to investigate an alternative Taylor-type interest rate rule,

\[ r_t = \varphi_\pi \pi^e_{t+1} \ + \ \varphi_x x^e_{t+1}, \]  

in which policymakers react to forecasts of future values of the inflation deviation and the output gap. Interest-rate rules depending on expectations of future inflation and the output gap have been discussed extensively in the monetary policy literature and are subject to various interpretations. Here we
are assuming that the monetary authorities form forecasts in the same way as the private sector, that is, by constructing an econometric forecast to which they consider adding the same judgement variable. We might hope that by reacting aggressively enough to expectations such a rule would diminish the likelihood of exuberance equilibria. With the policy rule (27) the reduced form system is the same as (26) except that

\[
\beta = \begin{bmatrix}
1 - \sigma^{-1}\varphi_x & \sigma^{-1}(1 - \varphi_\pi) \\
\kappa(1 - \sigma^{-1}\varphi_x) & \delta + \kappa\sigma^{-1}(1 - \varphi_\pi)
\end{bmatrix}
\]

and

\[
C = \begin{bmatrix}
1 & 0 \\
\kappa & 1
\end{bmatrix}.
\]

Using the same calibration, we calculate whether the conditions for exuberance equilibria hold for \(\varphi_x \in (0, 3.5)\) and \(\varphi_x \in (0, 0.35)\) at selected grid points. The results are plotted in Figure 4. The open squares again indicate the points where determinacy and learnability of the rational expectations equilibrium hold for this model.\(^{16}\) As with the standard Taylor-type rule, the figure indicates that exuberance equilibria exist near the point \((1, 0)\). Again, more aggressive policy delivers non-exuberance. Comparing this performance to that of the contemporaneous rule, we see that non-exuberance begins to arise for smaller values of \(\varphi_\pi\) and \(\varphi_x\) – see Figures 3 and 4. In particular, with the forward-looking rule even very small values of \(\varphi_x\) are sufficient to yield non-exuberance if \(\varphi_\pi\) is greater than (approximately) 1.8. In this sense the performance of the forward-looking rule appears superior, which provides one potential justification for their use by central banks.\(^{17}\) We conclude that by following an explicit policy of reacting against the deviations of expectations from the values justified by the fundamental shocks, monetary authorities enhance the stability of the economy.

---

\(^{16}\)For the forward-looking rule, indeterminacy of the fundamental rational expectations equilibrium occurs not only in the blank area to the left in the figure, but also in the blank area toward the top of the figure.

\(^{17}\)This example in Figure 4 also produces strong non-exuberance for large enough values of \(\varphi_x\), approximately 3.25 or greater in this figure, depending on the value of \(\varphi_\pi\).
Figure 4: Sufficiently aggressive policy is again associated with non-exuberance when the policy rule is forward-looking.

4.3.3 Optimal monetary policy rules

Finally, we discuss optimal discretionary policy as in Evans and Honkapohja (2003). They assign a standard quadratic objective to the policymaker with weight $\alpha$ on output gap variance. They write the resulting optimal policy as a Taylor-type rule in the expected output gap and the expected inflation deviation, along with reactions to fundamental shocks in the economy. Their policy rule delivers determinacy, and the unique stationary rational expectations equilibrium is stable under least squares learning for all values
of structural parameters and the policy weight. We can denote this optimal policy rule as

$$r_t = \varphi^*_\pi \pi_{t+1} + \varphi^*_x x_{t+1} + \varphi^*_{u,x} \bar{u}_{x,t} + \varphi^*_{u,\pi} \bar{u}_{\pi,t}.$$  (28)

where the optimal values $\varphi^*_x = \varphi^*_u = \sigma$, and the matrices $\beta$ and $C$ become

$$\beta = \begin{bmatrix} 0 & \sigma^{-1} (1 - \varphi^*_\pi) \\ 0 & \delta + \kappa \sigma^{-1} (1 - \varphi^*_\pi) \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & -\sigma^{-1} \varphi^*_u \\ 0 & 1 - \sigma^{-1} \kappa \varphi^*_u \end{bmatrix}.$$  

The relationship between $\varphi^*_u$ and $\varphi^*_\pi$ is given by $\varphi^*_u = \delta^{-1} (\varphi^*_\pi - 1)$. This leaves only the optimal choice of $\varphi^*_\pi$, which depends on $\alpha$, the policymaker weight on the output gap in the objective. A small weight on output gap variability $\alpha \to 0$ (an inflation hawk), is associated with an optimal value $\varphi^*_\pi = 1 + \sigma \delta \kappa^{-1} \approx 7.47$. A large weight on output gap variability, $\alpha \to \infty$, (an inflation dove), is associated with an optimal value $\varphi^*_\pi \to 1$. Thus we can calculate whether exuberance equilibria exist for all possible values of the policymaker weight $\alpha$ by choosing values for $\varphi^*_\pi \in (1, 7.47)$. 

The results of this calculation\(^{18}\) indicate that for values of $\varphi^*_\pi \in (1, \bar{\varphi})$ the equilibrium is in the indefinite region. For values $\varphi^*_\pi \in (\bar{\varphi}, 7.47)$, the equilibrium is non-exuberant. The cutoff value $\bar{\varphi}_\pi \approx 1.557$. Thus standard optimal policy calculations alone are not enough to ensure non-exuberance. To move into the non-exuberance region, policymakers must have a sufficiently small weight on output gap variability. The policy weight value associated with $\varphi^*_\pi = 1.557$ is quite low, approximately $\alpha \approx 0.00612$. More weight than this on output gap variance implies a value for $\varphi^*_\pi$ that is too low, in the sense that it places the equilibrium in the indefinite region.\(^{19}\)

\(^{18}\)The exact optimal policy rule would create perfect multicollinearity in this system. To avoid this complication, we set $\varphi_x = 1.01 \sigma$, slightly higher than the optimal value.

\(^{19}\)If we assume that the policymaker has the same preferences as the representative household, we obtain a value of $\alpha \approx .00313$ at the calibrated values of Woodford (2003). (This is calculated as $\kappa/\theta = 0.024/7.67$, where $\theta$ is the parameter controlling the price elasticity of demand.) The value of $\varphi^*_\pi$ for arbitrary $\alpha$ is $1 + \kappa \sigma (\alpha + \kappa^2)$. This would suggest an optimal value of $\varphi^*_\pi \approx 2.0$, large enough to imply non-exuberance.
5 Conclusions and possible extensions

We have studied how a new phenomenon, *exuberance equilibria*, may arise in standard macroeconomic environments. We assume that agents are learning in the sense that they are employing econometric models to forecast the future values of variables they care about. Unhindered, this learning process would converge to a rational expectations equilibrium in the economies we study. We investigate the idea that decision-makers may be tempted to include judgemental adjustments to their forecasts if all others in the economy are similarly judgementally adjusting their forecasts. The judgemental adjustment, or add factor, is a pervasive and widely-acknowledged feature of actual macroeconometric forecasting in industrialized economies. We obtain conditions under which such add-factoring can become self-fulfilling, altering the actual dynamics of the economy significantly, but in a way that remains consistent with the econometric model of the agents.

In order to develop our central points we have made some strong simplifying assumptions. We have assumed that the exuberance or judgement variables take a simple autoregressive form, but this assumption is mainly made for convenience. While we do believe that judgemental adjustments exhibit strong positive serial correlation, a more complicated stationary stochastic process could instead be used and in principle even time varying distributions could be incorporated into our framework.

The incorporation of judgment into decisions, in the form of adjustments to econometric forecasts, can have a self-fulfilling feature in the sense that decisions makers would believe *ex post* that their judgement had improved their forecasts. This result is similar in spirit to the self-fulfilling nature of sunspot equilibria, but with the novel feature that it can arise in determinate models in which there is a unique rational expectations equilibrium that depends only on fundamentals. In particular, we show that exuberance equilibria can arise in the standard asset-pricing model, generating substantial excess volatility. Exuberance equilibria can also arise in New Keynesian models, with monetary policymakers following standard interest-rate rules, but can
be eradicated if policymakers take an appropriately aggressive stance.

Our results may appear to be in sharp contrast to those of Svensson (2003, 2004), who shows how the use of judgement by policymakers can improve economic performance. However, our assumptions concerning judgement are very different. In Svensson, judgements are central bank forecasts of fundamental variables that have a direct effect on the economy. In contrast, we show the potential pitfalls in the use of judgement by economic agents when the judgement concerns phenomena that are believed to affect the economy, but in actuality do so only through expectational effects. A natural question for future research concerns the implications of incorporating judgement variables that are imperfectly correlated with fundamentals.
Appendices

A Conditions for CEE in the scalar case

The sum of the two functions $G_\eta (z)$ and $G_u (z)$ is

$$G_{ALM} (z) = \frac{(1 - az) (1 - az^{-1})}{(1 - \rho z) (1 - \rho z^{-1})} \times \left\{ \frac{\beta^2 \sigma_n^2 + (1 - \rho z) (1 - \rho z^{-1}) \sigma_u^2}{[\beta (a - b) + 1 - az] [\beta (a - b) + 1 - az^{-1}]} \right\}.$$

It can be seen from the form of $G_{ALM} (z)$ that, for arbitrary $a$ and $b$, the ALM is an ARMA(2,2) process. As we will now show, there are choices of $a$ and $b$ that yield $G_{PLM} (z) = G_{ALM} (z)$. These choices of $a$ and $b$ also have the property that the corresponding ALM takes an ARMA(1,1) form that matches the PLM. This is possible if $a$ and $b$ are chosen so that there is a common factor in the numerator and denominator of the expression on the right-hand side of $G_{ALM} (z)$.

We now set $G_{PLM} (z) = G_{ALM} (z)$, under the condition that $b = \rho$ so that the poles of the autocovariance generating functions agree. This yields

$$\sigma_v^2 [\beta (a - \rho) + 1 - az] [\beta (a - \rho) + 1 - az^{-1}] = \beta^2 \sigma_n^2 + (1 - \rho z) (1 - \rho z^{-1}) \sigma_u^2.$$

This equation can be written as

$$\sigma_v^2 \{[1 + \beta (a - \rho)]^2 + a^2\} - \sigma_n^2 a [\beta (a - \rho) - 1] (z + z^{-1}) = \beta^2 \sigma_n^2 + \sigma_u^2 (1 + \rho^2) - \sigma_u^2 \rho (z + z^{-1}).$$

For the autocovariances of the perceived and actual laws of motion to be equal, the coefficients on the powers of $z$ in this equation must be equal. By equating coefficients on powers of $z$, we obtain two equations, which are given by

$$\sigma_v^2 \{[1 + \beta (a - \rho)]^2 + a^2\} = \beta^2 \sigma_n^2 + \sigma_u^2 (1 + \rho^2) \quad (29)$$
and
\[ \sigma_v^2 a [\beta (a - \rho) + 1] = \sigma_u^2 \rho. \]  \hspace{1cm} (30)
We wish to solve for a value of \( a \) such that \(|a| < 1\). Solving equation (30) for \( \sigma_v^2 \) and substituting the result into equation (29), and in addition defining
\[ s \equiv \beta^2 \sigma_v^2 + \sigma_u^2 (1 + \rho^2), \]
we obtain the quadratic equation
\[ f(a) \equiv c_2 a^2 + c_1 a + c_0 = 0 \] \hspace{1cm} (31)
with
\[ c_2 \equiv s \beta - \rho (1 + \beta^2) \sigma_u^2, \]
\[ c_1 \equiv s (1 - \rho \beta) - 2 \rho \beta (1 - \rho \beta) \sigma_u^2, \]
\[ c_0 \equiv -\rho (1 - \rho \beta)^2 \sigma_u^2. \]
We deduce that \( f(0) < 0 \), and that
\[ f(1) = \sigma_v^2 \beta^2 \left[ 1 + (1 - \rho) \beta \right] + \sigma_u^2 (1 - \rho \beta) (1 + \beta) \left[ (\rho - 1)^2 \right] > 0. \]
These inequalities imply that there exists a positive root \( a \in [0, 1] \) to (31). Moreover, it is easy to compute that \( f(\rho) > 0 \), so that the root must be less than \( \rho \). We also note that for \( \sigma_v^2 \to 0 \), \( a = \rho \) solves equation (31), while for \( \sigma_u^2 \to 0 \), \( a = 0 \) is a solution. There can be a second, negative root. However, our numerical results indicate that the corresponding CEE is not learnable.

B  Judgement in the scalar case

B.1  Special case

The induced actual law of motion, as depicted in equation (14), is
\[ y_t = \frac{1 - aL}{\beta (a - \rho) + 1 - aL} \bigg( \frac{\beta}{1 - \rho L} \eta_t + u_t \bigg). \] \hspace{1cm} (32)
By substituting equation (32) into both (16) and (17), we can write the two types of forecasts in terms of the shocks \( u_t \) and \( \eta_t \). These expressions become
\[ E^*_t y_{t+1} = \frac{\rho - a}{\beta (a - \rho) + 1 - aL} \bigg( \frac{\beta}{1 - \rho L} \eta_t + u_t \bigg) \]

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in the case of no judgement, and
\[ y_{t+1}^e = \frac{\rho - a}{\beta (a - \rho) + 1 - aL} \left( \frac{\beta}{1 - \rho L} \eta_t + u_t \right) + \frac{1}{1 - \rho L} \eta_t \]
in the case of the judgementally adjusted forecast. The actual state of the economy at time \( t + 1 \) is, from equation (32),
\[ y_{t+1} = \frac{1 - aL}{\beta (a - \rho) + 1 - aL} \left( \frac{\beta}{1 - \rho L} \eta_{t+1} + u_{t+1} \right). \] (33)

We can therefore compute forecast errors in each of the two cases. When computing these forecast errors, we save on clutter by ignoring the terms involving \( u \), as these will be the same whether or not the agent judgementally adjusts the forecast. The forecast error in the case of no judgement can be written as
\[ FE_{NJ} \equiv \left[ y_{t+1} - E_t y_{t+1} \right]_{u=0} = \frac{\beta}{1 + \beta (a - \rho)} \left[ 1 - \left( \frac{a}{1 + \beta (a - \rho)} \right) L \right] \eta_{t+1} \] (34)
whereas in the case of a judgementally adjusted forecast it is
\[ FE_J \equiv \left[ y_{t+1} - y_{t+1}^e \right]_{u=0} = \frac{\beta}{1 + \beta (a - \rho)} \times \frac{1 - (a + \beta^{-1}) L + a \beta^{-1} L^2}{1 - \left( \frac{a + \beta (a - \rho)}{1 + \beta (a - \rho)} \right) L + \left( \frac{a \rho}{1 + \beta (a - \rho)} \right) L^2} \eta_{t+1}. \] (35)

These equations simplify to those given in the text when \( a \to \rho \).

**B.2 General case**

For a process in this class written generically as
\[ x_t = \frac{1 + \theta_1 L + \theta_2 L^2}{1 - \phi_1 L - \phi_2 L^2} \epsilon_t, \]
the variance of \( x_t \) is given by
\[ Var(x_t) = \frac{x_{num}}{x_{den}} \sigma^2. \] (36)
where
\[ x_{num} = \frac{(1 + \phi_2) \phi_1 (\theta_1 + \theta_2 \phi_1 + \theta_2 \phi_1)}{1 - \phi_2} + (\theta_1 + \theta_2 \phi_1) (\phi_1 + \theta_1) + (1 + 2 \theta_2 \phi_2 + \phi_2^2) \]
and
\[ x_{den} = 1 - \frac{\phi_1^2}{1 - \phi_2} - \frac{\phi_2 \phi_1^2}{1 - \phi_2} - \phi_2^2 \]

Considering the forecast error in the case without judgement included, equation (34), we set \( \theta_1 = \theta_2 = \phi_2 = 0 \) and \( \phi_1 = a / [1 + \beta (a - \rho)] \) in equation (36). For the case with judgement, we set
\[
\begin{align*}
\theta_1 &= -(1 + a \beta) \beta^{-1}, \\
\theta_2 &= a \beta^{-1}, \\
\phi_1 &= a + \rho [1 + \beta (a - \rho)] \frac{1 + \beta (a - \rho)}{1 + \beta (a - \rho)}, \\
\phi_2 &= -a \rho \\
&= 1 + \beta (a - \rho).
\end{align*}
\]
As we have seen, the value of \( a \) can be influenced independently of the values of \( \rho \) and \( \beta \) by choice of the relative variance \( R \).

C  Recursive learning

C.1  Recursive least squares

For simplicity, we develop the details in the univariate setting. Econometricians estimate the PLM
\[
y_t = \sum_{i=1}^{p} b_i y_{t-i} + v_t
\]
using recursive least squares. Let \( b_t = (b_{1,t}, ..., b_{p,t}) \) denote the parameter estimates at time \( t \) and let \( Y_{t-1}' = (y_{t-1}, ..., y_{t-p}) \) be the vector of state variables. The RLS algorithm is
\[
\begin{align*}
b_t' &= b_{t-1}' + t^{-1} R_{t-1}^{-1} Y_{t-1} (y_t - b_{t-1} Y_{t-1}) \\
R_t &= R_{t-1} + t^{-1} (Y_{t-1} Y_{t-1}' - R_{t-1}),
\end{align*}
\]
where \( y_t \) is given by the ALM (20) with \( b_t \) replaced by \( b_{t-1} \). Here \( R_t \) is an estimate of the matrix of second moments of \( Y_{t-1} \) and the first equation is just the recursive form of the usual least squares formula. Note that assumptions about timing are as follows. At the end of period \( t-1 \) econometricians update their parameter estimates to \( b_{t-1} \) using data up to \( t-1 \). At time \( t \) econometricians use these parameter estimates and observed \( Y_t \) to make their forecast \( E_t^* y_{t+1} \). At the end of time \( t \) econometricians update the parameters to \( b_t \). For further discussion of RLS learning see Chapters 2 and 8 of Evans and Honkapohja (2001).

The question of interest is whether \( \lim_{t \to \infty} b_t \to \bar{b} \), where \( \bar{b} = (\bar{b}_1, \ldots, \bar{b}_p) \) denotes the approximate CEE. In this case \( \bar{b} \) is said to be locally learnable. It can be shown that the asymptotic dynamics of \( (b_t', R_t) \) are governed by an associated differential differential equation and that, in particular, the asymptotic dynamics of \( b_t \) are governed by

\[
\frac{db}{d\tau} = \left[ E y_t(b) Y_{t-1}(b)' \right] \left[ E Y_t(b) Y_{t-1}(b)' \right]^{-1} - b = T(b) - b.
\]

Here \( \tau \) denotes notional or virtual time, \( y_t(b) \) is the stationary stochastic process given by (20) for fixed \( b \) and \( Y_{t-1}(b)' = (y_{t-1}(b), \ldots, y_{t-p}(b)) \). Numerically, convergence can be verified using the E-stability algorithm (22), which can also be used to compute the approximate CEE.

The above procedure can easily be generalized to the multivariate case in which the PLM is a VAR(\( p \)) process.

### C.2 Recursive maximum likelihood

We now consider recursive estimation when the PLM is an ARMA(1,1) process, i.e.

\[
y_t = by_{t-1} + v_t + cv_{t-1},
\]

where \( y_t \) is observed but the white noise process \( v_t \) is not observed. Let \( b_t \) and \( c_t \) denote the estimates of \( b \) and \( c \) using data through time \( t-1 \). The

\footnote{Note that \( y_t \) and \( E_t^* y_{t+1} \) are simultaneously determined. Alternative information assumptions could be made but would not affect our main results.}
econometricians are assumed to use a recursive maximum likelihood (RML) algorithm, which we now describe. \(^{21}\)

Let \(\phi_t = (b_t, c_t)\). To implement the algorithm an estimate \(\varepsilon_t\) of \(v_t\) is required. Let \(\varepsilon_t = y_t - x_{t-1}^T \phi_{t-1}\), where \(x_{t-1} = (y_{t-1}, \varepsilon_{t-1})\). \(y_t\) is given by \(y_t = \beta [E_t^* y_{t+1} + \xi_t] + u_t\), where \(E_t^* y_{t+1} = b_{t-1} y_t + c_{t-1} \varepsilon_t\). The RML algorithm is as follows

\[
\begin{align*}
\psi_t &= -c_{t-1} \psi_{t-1} + x_t \\
\phi_t &= \phi_{t-1} + t^{-1} R_{t-1}^{-1} \psi_{t-1} \varepsilon_t \\
R_t &= R_{t-1} + t^{-1} (\psi_{t-1} \psi_{t-1}^T - R_{t-1})
\end{align*}
\]

Again the question of interest is whether \(\phi_t\) converges to an exact CEE. Convergence can be studied using the associated ordinary differential equation

\[
\begin{align*}
\frac{d\phi}{d\tau} &= R^{-1} E_{\psi_t}^T(\phi) \varepsilon_t(\phi) \quad (37) \\
\frac{dR}{d\tau} &= E_{\psi_t}^T(\phi) \psi_t(\phi)' - R. \quad (38)
\end{align*}
\]

Here \(y_t(\phi), \psi_t(\phi)\) and \(\varepsilon_t(\phi)\) denote the stationary processes for \(y_t, \psi_t\) and \(\varepsilon_t\) with \(\phi_t\) set at a constant value \(\phi\). Using the stochastic approximation tools discussed in Marcel and Sargent (1989), Evans and Honkapohja (1998) and Chapter 6 of Evans and Honkapohja (2001), it can be shown that the RML algorithm locally converges provided the associated ordinary differential equation is locally asymptotically stable (analogous instability results are also available). Numerically, convergence of (37)-(38) can be verified using a discrete time version of the differential equation. A first order state space form is convenient for computing the expectations \(E_{\psi_t}^T(\phi) \varepsilon_t(\phi)\) and \(E_{\psi_t}^T(\phi) \psi_t(\phi)'\) and this procedure was used for the numerical illustrations given in the main text.

We now prove convergence analytically for all \(0 < \beta, \rho < 1\) with \(\sigma_{\eta}^2 > 0\) sufficiently small. This completes the proof of part (i) in Theorem 1. We

\(^{21}\)For further details on the algorithm see Section 2.2.3 of Ljung and Soderstrom (1983). The algorithm is often called a recursive prediction error algorithm.
rewrite the system (37)-(38) in the form
\[
\frac{d\phi}{d\tau} = (R)^{-1}f(\phi)
\]
\[
\frac{dR}{d\tau} = M_\psi(\phi) - R
\]
where we have introduced the simplifying notation \(f(\phi) = E\psi_t(\phi)\xi_t(\phi)\) and \(M_\psi(\phi) = E\psi_t(\phi)\psi_t(\phi)'\). An equilibrium \(\bar{\phi}, \bar{R}\) of the system is defined by \(f(\bar{\phi}) = 0\) and \(\bar{R} = M_\psi(\bar{\phi})\). As discussed in the main text, there can be two equilibrium values \(\bar{\phi}' = (\rho, -a)\) determined by the solutions to the quadratic (31), but we here focus on the solution with \(0 < a < 1\). Recall that for this solution \(a \to \rho\) as \(\sigma_\eta^2 \to 0\).

Linearizing the system at the equilibrium point, it can be seen that the linearized system has a block diagonal structure, in which one block has the eigenvalues equal to \(-1\) (with multiplicity four) and the eigenvalues of the other block are equal to those of the “small” differential equation
\[
\frac{d\phi}{d\tau} = (\bar{R})^{-1}J(\bar{\phi})(\phi - \bar{\phi}),
\]
where \(J(\phi)\) is the Jacobian matrix of \(f(\phi)\). The system (37)-(38) is therefore locally asymptotically stable if the coefficient matrix \((\bar{R})^{-1}J(\bar{\phi})\) of the two-dimensional linear system (39) has a negative trace and a positive determinant. Since \((\bar{R})^{-1} = (\det(\bar{R}))^{-1}\text{adj}(\bar{R})\) we have
\[
\begin{align*}
\text{Tr}[(\bar{R})^{-1}J(\bar{\phi})] &= (\det(\bar{R}))^{-1}\text{Tr}[\text{adj}(\bar{R})J(\bar{\phi})] \\
\det[(\bar{R})^{-1}J(\bar{\phi})] &= \det[(\bar{R})^{-1}]\det[J(\bar{\phi})].
\end{align*}
\]
Now \(\det(\bar{R}) > 0\) as \(\bar{R}\) is a matrix of second moments and thus positive definite for \(\sigma_\eta^2 > 0\). It thus remains to prove that \(\text{Tr}[\text{adj}(\bar{R})J(\bar{\phi})] < 0\) and \(\det[J(\bar{\phi})] > 0\) when \(\sigma_\eta^2 > 0\) is sufficiently small.

We consider the values of \(\text{Tr}[\text{adj}(\bar{R})J(\bar{\phi})]\) and \(\det[J(\bar{\phi})]\) when \(\sigma_\eta^2 \to 0\). Using the definition of \(\xi_t\), the explicit form of \(f(\phi)\) is
\[
f(\phi) = E\psi_{t-1}(\phi)x_{t-1}' \left[ (1 - \beta b - \beta c)^{-1} \beta \begin{pmatrix} -bc \\ -c^2 \end{pmatrix} - \begin{pmatrix} b \\ c \end{pmatrix} \right] + (1 - \beta b - \beta c)^{-1} \beta \rho E\psi_{t-1}(\phi)\xi_{t-1},
\]

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where the moment matrices $E\psi_{t-1}(\phi)x'_{t-1}$ and $E\psi_{t-1}(\phi)\xi_{t-1}$ can be computed from the state space form

$$AX_t = CX_{t-1} + F \begin{bmatrix} u_t \\ \eta_t \end{bmatrix}, \text{ with } X_t = \begin{pmatrix} y_t \\ \varepsilon_t \\ \xi_t \\ \psi_t \\ \psi_{t-1} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & -c \end{pmatrix},$$

$$F = \begin{pmatrix} (1 - \beta b)^{-1} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be computed using Mathematica (routine available on request) that $Tr[adj(\bar{R})J(\bar{\phi})]$ and $det[J(\bar{\phi})]$ have the following properties as functions of (using temporary notation) $\omega \equiv \sigma^2_{\eta}$:

$$\lim_{\omega \to 0} Tr[adj(\bar{R})J(\bar{\phi})] = \lim_{\omega \to 0} \frac{d}{d\omega} Tr[adj(\bar{R})J(\bar{\phi})] = 0,$$

$$\lim_{\omega \to 0} det[J(\bar{\phi})] = \lim_{\omega \to 0} \frac{d}{d\omega} det[J(\bar{\phi})] = 0,$$

$$\lim_{\omega \to 0} \frac{d^2}{d\omega^2} Tr[adj(\bar{R})J(\bar{\phi})] = \frac{4\beta^4 \rho^2}{(1 - \beta \rho)(\rho^2 - 1)^6} < 0 \text{ and}$$

$$\lim_{\omega \to 0} \frac{d^2}{d\omega^2} det[J(\bar{\phi})] = \frac{2\beta^4 \rho^2}{(\rho^2 - 1)^6} > 0.$$

Expressing $Tr[adj(\bar{R})J(\bar{\phi})]$ and $det[J(\bar{\phi})]$ in terms of Taylor series these results show that

$$Tr[adj(\bar{R})J(\bar{\phi})] < 0 \text{ and } det[J(\bar{\phi})] > 0$$

for $\sigma^2_{\eta} > 0$ sufficiently small. $Q.E.D.$
References


